UNIVERSITY OF LINCOLN
SCHOOL OF MATHEMATICS AND PHYSICS

MTH1005M PROBABILITY AND STATISTICS
PRACTICAL 3

TOPICS: Calculation of probabilities of events occurring in simple sample spaces with equally likely outcomes, conditional probabilities.

Question 1

Two fair six sided dice are thrown

- the first die comes down a 6, what is the chance the total is greater than 10?
- it is known that at least one of the dice shows > 3. Find the probability that at least one of them is a six.

CONDITIONAL PROBABILITY

Definition : the conditional probability of the event E given the event F is

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

combined with the standard properties of probabilities this implies

$$P(\bar{E}|F) = 1 - \frac{P(E \cap F)}{P(F)}$$

if E and F are mutually exclusive, then $P(E|F) = 0$ because $E \cap F = \emptyset$.

Question 1

Two fair six sided dice are thrown

- the first die comes down a 6, what is the chance the total is greater than 10?
- it is known that at least one of the dice shows > 3. Find the probability that at least one of them is a six.

Solution

- We have two events - E_1 = "the first die shows 6 spots" and E_2 = "the total is greater than 10". By drawing a table or enumeration

$$E_1 = \{(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\} \tag{0.1}$$

$$\tag{0.2}$$

$$E_2 = \{(5,6),(6,5),(6,6)\} \tag{0.3}$$

Each outcome has equal likelihood if the dice are fair, so we can work out probabilities from the number of elements in suitable sets. We are looking for $P(E_2|E_1)$

$$P(E_2 \mid E_1) = \frac{P(E_2 \cap E_1)}{P(E_1)} = \frac{n(E_2 \cap E_1)}{n(E_1)} = \frac{2}{6}$$

Question 1

Two fair six sided dice are thrown

- the first die comes down a 6, what is the chance the total is greater than 10?
- it is known that at least one of the dice shows > 3. Find the probability that at least one of them is a six.

- We have two events:

  $E_1 =$ "at least one die shows great than 3 spots"

and

$E_2 =$ "at least one of the dice is a six". By drawing a table or enumeration

$$E_1 = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3,(3,1),(3,2),(3,3)\}^C \tag{0.4}$$

$$\tag{0.5}$$

$$E_2 = \{(1,6),(2,6),(3,6),(4,6),(5,6),(6,6),(6,1),(6,2),(6,3),(6,4),(6,5)\} \tag{0.6}$$

$$P(E_2 \mid E_1) = \frac{P(E_2 \cap E_1)}{P(E_1)} = \frac{n(E_2 \cap E_1)}{n(E_1)} = \frac{n(E_2)}{n(E_1)} = \frac{11}{27}$$

We used the fact that $E_2 \subset E_1$ in the last but one step.

Question 2

Three letters are chosen at random from all the letters of the word AEGEAN. Find the probability that

- the first letter is a consonant,
- either the second or the third letter is a vowel,
- an 'E' is not included.

Hint - it may be easier to work with complements - i.e. events that don't include some outcome

MULTIPLICATIVE RULE OF PROBABILITIES

The probability that E and F occur is the probability that F occurs multiplied by theăprobability that E occurs given that F has occurred.

$$P(E \cap F) = P(E|F)P(F)$$

and for many events

$$P(E_1 \cap E_2 \cap E_3 \cap \ldots E_n) = P(E_1)P(E_2|E_1)P(E_3|E_1 \cap E_2)\ldots P(E_n|E_1 \cap \ldots E_{n-1})$$

Question 2

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SOLUTIONS

Define the events

A_i ="any letter as the ith selection",

V_i ="a vowel in ith place" and

C_i = "a consonant in the ith location".

$P(C_1)$ is just the number of consonants / number of total letters = $\frac{2}{6}$

Either the second or third is a vowel is the same as not both being consonants.

So $P(V_2 \cup V_3)$ is the same as $1 - P(A_1 C_2 C_3)$,
$P(C_1 C_2 C_3 = 0$ as there are only two consonants so, using the multiplicative law of probabilities:

$$P(V_2 \cup V_3) = 1 - P(V_1) \cdot P(C_2 / V_1) \cdot P(C_3 / V_1 \cap C_2) = 1 - \frac{4}{6} \cdot \frac{2}{5} \cdot \frac{1}{4} = \frac{14}{15}$$

P('no letter Es selected') $= P(\bar{E_1} \cap \bar{E_2} \cap \bar{E_3}))$, where E_i = 'E as the ith letter', so $\bar{E_i}$ is the event that a letter 'E' is not selected on the ith selection. Then we can use the multiplicative law again

$$P(\bar{E_1} \cap \bar{E_2} \cap \bar{E_3})) = P(\bar{E_1}) \cdot P(\bar{E_2} / \bar{E_1}) \cdot P(\bar{E_3} / \bar{E_1} \cap \bar{E_2}) = \frac{4}{6} \cdot \frac{3}{5} \cdot \frac{2}{4} = \frac{1}{5}$$

Question 3

In a certain college, 4% of the men and 1% of the women are taller than 6 feet. Furthermore, 60% of the students are women. Now if a student is selected at random and is taller than 6 feet, what is the probability that the student is a woman?

BAYES' FORMULA

likelihood

Prior Probability

$$P(E_i \mid A) = \frac{P(A \mid E_i)P(E_i)}{P(A)}$$

Marginal probability

Posterior probability

LAW OF TOTAL PROBABILITY

Given a set of exaustive event $E_i \in S$. The total probability for the event $A \cap E_i$ is given by

$$P(A) = \sum^{n} P(A \cap E_i) = \sum_{i=1}^{n} P(A|E_i)P(E_i)$$

BAYES' FORMULA

$$P(E_i|A) = \frac{P(A \cap E_i}{P(A)} = \frac{P(A|E_i)P(E_i)}{\sum_{i=1}^{n} P(A|E_i)P(E_i)}$$

with

- $P(E_i|A)$: the posterior probability.
- $P(A|E_i)$: likehood.
- $P(E_i)$: prior probability.
- $P(A)$: total probability (or marginal probability).

Question 3

In a certain college, 4% of the men and 1% of the women are taller than 6 feet. Furthermore, 60% of the students are women. Now if a student is selected at random and is taller than 6 feet, what is the probability that the student is a woman?

Solution

Let A = *{students taller than 6 feet}*.
We seek $P(W|A)$, the probability that a student is a woman given that the student is taller than 6 feet.
We have that

$$P(W) = 0.6;$$

$$P(M) = 0.4;$$

$$P(A|M) = 0.04;$$

$$P(A|W) = 0.01;$$

By Bayes' theorem,

$$P(W|A) = \frac{P(A|W)P(W)}{P(A|W)P(W) + P(A|M)P(M)} = \frac{(0.01)(0.60)}{(0.01)(0.60) + (0.04)(0.40)} = \frac{3}{11}$$

Question 4

Six married couples are standing in a room.

1. If 2 people are chosen at random, find the probability p that (a) they are married, (b) one is male and one is female.

2. If 4 people are chosen at random, find the probability p that (a) 2 married couples are chosen, (b) no married couple is among the 4, (c) exactly one married couple is among the 4.

Combinations. The number of way to combine n distinguishable objects from a group of $N \leq n$ distinguishable objects is

$$c(n,r) = \binom{N}{k} = \frac{N!}{(N-n)!n!}$$

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2. If 4 people are chosen at random, find the probability p that (a) 2 married couples are chosen, (b) no married couple is among the 4, (c) exactly one married couple is among the 4.

Solution

1. There are $\binom{12}{2} = 66$ ways to choose 2 people from the 12 people.

   (a) There are 6 married couples; hence $p = \frac{6}{66} = \frac{1}{11}$.

   (b) There are 6 ways to choose a male and 6 ways to choose a female; hence $p = \frac{6\times 6}{66}$.

2. There are $\binom{12}{4} = 495$ ways to choose 4 people from the 12 people.

   (a) There are $\binom{6}{2} = 15$ ways to choose 2 couples from the 6 couples; hence $p = \frac{15}{495} = \frac{1}{33}$.

   (b) The 4 persons come from 4 different couples. There are $\binom{6}{4} = 15$ ways to choose 4 couples from the from the 6 couples, and there are 2 ways to choose one person from each couple. Hence $\frac{2\times 2\times 2\times 2\times 15}{495} = \frac{16}{33}$.

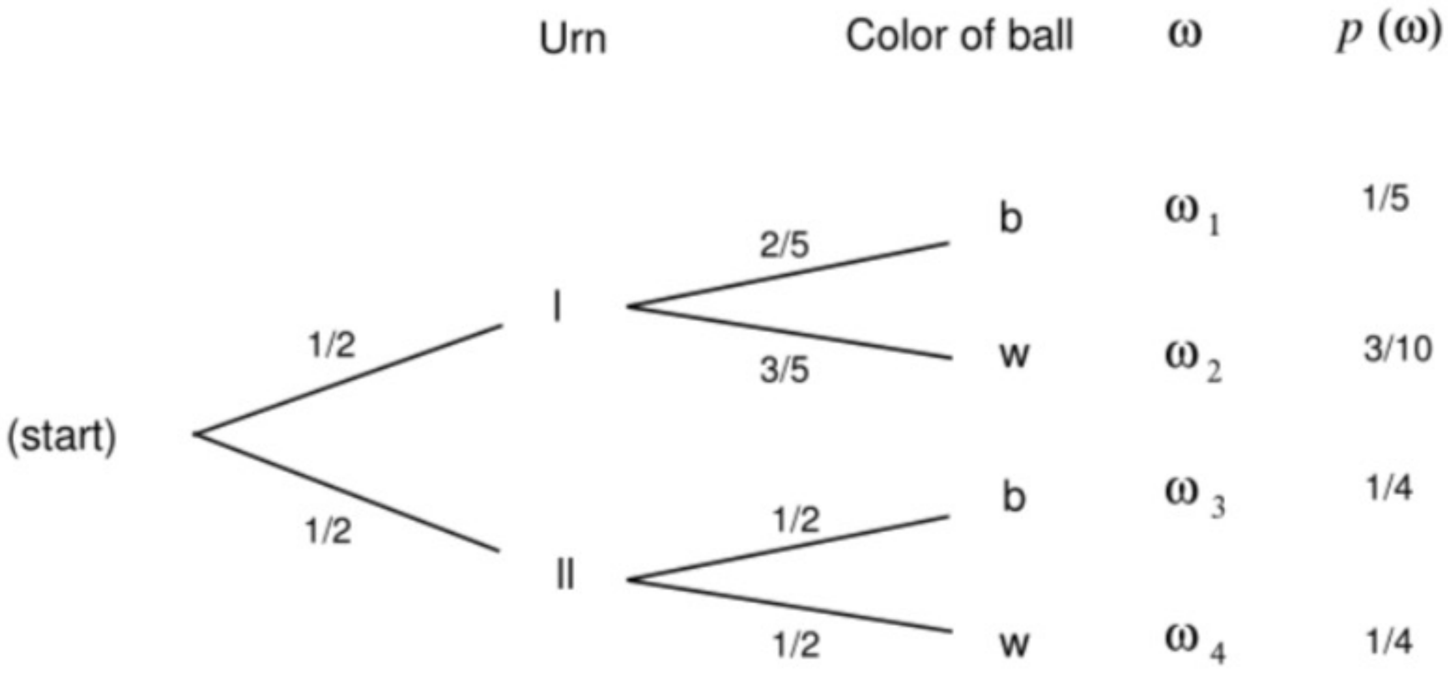
   (c) This event is mutually disjoint from the preceding two events (which are also mutually disjoint) and at least one of these events must occur. Hence $p + \frac{1}{33} + \frac{16}{33} = 1$ or $p = \frac{16}{33}$.

TREE DIAGRAMS

- Tree diagrams are a useful way of keeping track of the progress of compound experiments.
- They can also be seen to work using the multiplicative rule of probabilities.
- Let's analyse throwing 3 coins sequentially.

TREE DIAGRAMS

We can present the sample space of this experiment as the paths through a tree as shown



Tree diagram.

The probabilities assigned to the paths are also shown.

Question 5

In Glasgow, half of the days have some rain. The local weather forecaster is correct $\frac{2}{3}$ of the time, i.e., the probability that she has predicted rain on a rainy day, and the probability that she predicts no rain on a dry day, are both equal to $\frac{2}{3}$.
When rain is forecast, a Glaswegian lady takes her umbrella. When rain is not forecast, she takes it with probability $\frac{1}{3}$.

- Draw a tree diagram - start labelling by whether it rains, then what the forecaster will predict and finally what the Glaswegian lady does.
- Check you know what types of probability you are labelling on the diagram.
- Calculate the probability that the lady has no umbrella, given that it rains.
- Calculate the probability that she brings her umbrella, given that it doesn't rain.

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- Draw a tree diagram - start labelling by whether it rains, then what the forecaster will predict and finally what the Glaswegian lady does.
- Check you know what types of probability you are labelling on the diagram.

Solution

The English word given is a strong clue that we need a conditional probability. I couldn't easily do this is my head, but needed to draw up a tree diagram:

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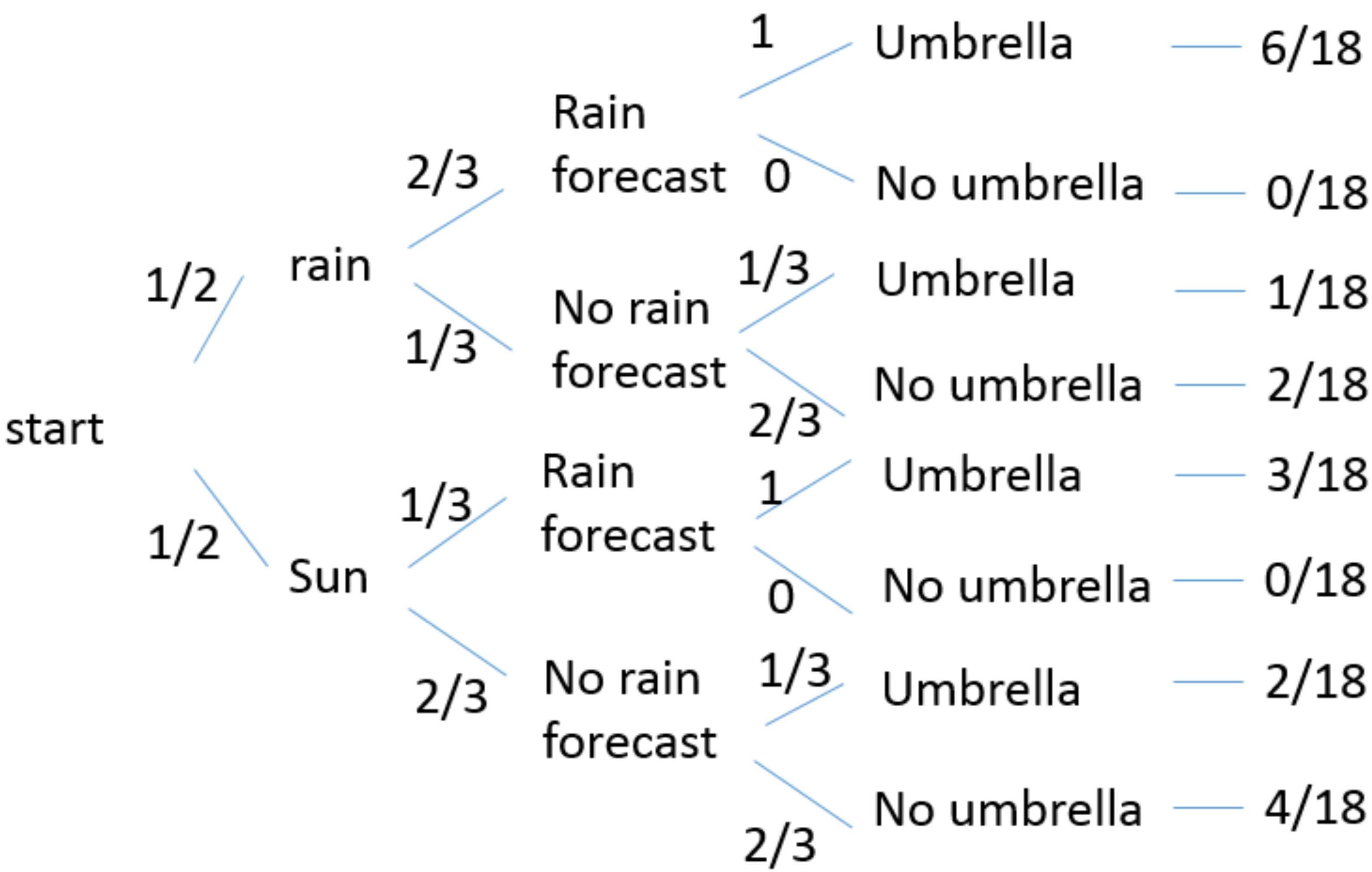

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combined with the standard properties of probabilities this implies

$$P(\bar{E}|F) = 1 - \frac{P(E \cap F)}{P(F)}$$

if E and F are mutually exclusive, then $P(E|F) = 0$ because $E \cap F = \emptyset$.

- $P(\text{'The lady has no umbrella | it rains'}) = \frac{P(\text{No umbrella} \cap \text{rain})}{P(\text{no rain})} = \frac{\frac{2}{18}}{\frac{1}{2}} = \frac{2}{9}$

- $P(\text{'The lady has an umbrella | it doesn't rain'}) = \frac{P(\text{Umbrella} \cap \text{no rain})}{P(\text{rain})} = \frac{\frac{3}{18}+\frac{2}{18}}{\frac{1}{2}} = \frac{5}{9}$

Question 6

One bag contains two dice:

- a fair one that gives equiprobable outcomes of 1 to 6.
- a "rigged" one: throwing it always gets 6.

A player has rolled one of the dice and, without examining it, rolled it (once) to get a 6. What is the probability that he used the loaded die?

Solution

Let

$$H = \text{"the loaded die has been rolled"}$$

$$K = \text{"the regular die has been rolled"}$$

$$A = \text{"came out on 6"}$$

,
it's $H \cup K = S$ and $H \cap K = \emptyset$.
We want to calculate $P(H|A)$, i.e. the probability that the tricked die was used, KNOWING that 6 came out. It is a simple application of Bayes' formula. In our example it is:

$$P(A) = P(A|H)P(H) + P(A|K)P(K)$$

and

$$P(H|A)P(A) = P(A|H)P(H) = P(A \cap H)$$

therefore

$$P(H|A) = \frac{P(A|H)P(H)}{P(A|H)P(H) + P(A|K)P(K)}$$

Since

$$P(A|H) = 1;$$

$$P(H) = P(K) = \frac{1}{2};$$

$$P(A|K) = \frac{1}{6},$$

we obtain

$$P(H|A) = \frac{6}{7}$$

A comparison between $P(H)$ and $P(H|A)$ shows how much the hypothesis H is supported by the information "6 came out" (it goes from 50% to about 86%).

Question 7

Microchips are made by three companies. 30% are supplied by firm I, 50% by II and 20% by III.
The probabilities of $A =$ *'a defect in a chip'* are $P(A|I) = 0.03$, $P(A|II) = 0.04$, $P(A|III) = 0.01$.
If an unlabeled box of chips turns up, what are the probabilities that the box came from I, II or III

(i) if a random test shows a defective chip?

(ii) if a random test shows a non-defective chip?

(iii) if the first chip was defective, and a second chip was then also tested and found defective?

Question 7

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The probabilities of $A =$ *'a defect in a chip'* are $P(A|I) = 0.03, P(A|II) = 0.04, P(A|III) = 0.01$.
If an unlabeled box of chips turns up, what are the probabilities that the box came from I, II or III

(i) if a random test shows a defective chip?

Solution

(i) We need to use Bayes' theorem - we are after the conditional probabilites $P(I|A), P(II|A), P(III$
Therefore

$$P(I|A) = \frac{P(I \cap A)}{P(A)}$$

$$= \frac{P(I)P(A|I)}{P(I)P(A|I) + P(II)P(A|II) + P(III)P(A|III)}$$

$$= \frac{(0.3)(0.03)}{(0.3)(0.03) + (0.5)(0.04) + (0.2)(0.01)} = \frac{9 \times 10^{-3}}{0.031} = 0.290$$

similarly, $P(II|A) = 0.645$ and $P(III|A) = 0.064$.

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The probabilities of $A =$ *'a defect in a chip'* are $P(A|I) = 0.03, P(A|II) = 0.04, P(A|III) = 0.01$.
If an unlabeled box of chips turns up, what are the probabilities that the box came from I, II or III

(i) if a random test shows a defective chip?

(ii) if a random test shows a non-defective chip?

(ii) This time we should work with probabilities of $\bar{A}$,

$$P(I|\bar{A}) = \frac{P(I \cap \bar{A})}{P(\bar{A})}$$

$$= \frac{P(I)P(\bar{A}|I)}{P(I)P(\bar{A}|I) + P(II)P(\bar{A}|II) + P(III)P(\bar{A}|III)}$$

$$= \frac{(0.3)(0.97)}{(0.3)(0.97) + (0.5)(0.96) + (0.2)(0.99)} = \frac{(0.3)(0.97)}{0.969} = 0.300$$

similarly, $P(II|\bar{A}) = 0.495$ and $P(III|\bar{A}) = 0.204$.

(iii) if the first chip was defective, and a second chip was then also tested and found defective?

(iii) After the first sample we have our revised (posterior) probabilities - $P(I|A), 0.290, P(II|A) = 0.645$ and $P(II|A) = 0.064$. We can take these as our new $P(I), P(II), P(III)$ and apply Bayes' theorem again (this was just mentioned in the lecture). In the second round we get

$$P(I|A) = \frac{(0.290)(0.03)}{(0.290)(0.03) + (0.645)(0.04) + (0.064)(0.02)} = \frac{(0.290)(0.03)}{(0.0358} = 0.243$$

similarly, $P(II|A) = 0.720$ and $P(III|A) = 0.036$.

We can look at this as using the information received (i.e. the chips were defective) to gain extra insight into the probabilities.
As we continue to sample, the posterior probability distribution will move to represent the distribution of chips in the box, rather than our initial guess, which is based on the number of chips each supplier supplies.

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MTH1005M PROBABILITY AND STATISTICS
PRACTICAL 4

Short review

Definition 1: Consider an experiment, with outcome set S, split into n mutually exclusive and exhaustive events $E_1, E_2, E_3, \ldots, E_n$. A variable, X say, which can assume exactly n numerical values each of which corresponds to one and only one of the given events is called a random variable.

Definition 2: Let X be a discrete random variable that can take on only the values $x_1, x_2, x_3, \ldots, x_n$ with respective probabilities $p_X(x_1), p_X(x_2), p_X(x_3), \ldots, p(x_n)$.
Then, if

$$\sum_{x \in X} p_X(x) = 1$$

The function $p(x) = P\{X = x\}$ is called the probability (mass) function of the variable X.

Definition 3: The cumulative distribution function of a discrete random variable, X, is

$$F_X(a) = P\{X \leq a\} = \sum_{x \leq a} p_X(x).$$

X is a discrete random variable.

STRING LOGIC

Check list

- is the sample space well-defined?
- are the events mutually exclusive?
- do the events cover all the sample space?
- are the probabilities of the events defined?
- are values of the variable assigned one-to-one to the possible events?

Question 1

Let Z be the 'number around the base when a 4 sided die is rolled'.

- Write down a sample space for the experiment
- Define the values of Z and the events they correspond to.
- Write down the probability mass function of Z.

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Let Z be the 'number around the base when a 4 sided die is rolled'.

- Write down a sample space for the experiment
- Define the values of Z and the events they correspond to.
- Write down the probability mass function of Z.

Solution

- The sample space is a set of all possible outcomes $S = \{1, 2, 3, 4\}$
- $Z = 1$ if '1s are around the base of the die', or $\{1\}$
  $Z = 2$ if '2s are around the base of the die', or $\{2\}$

  $Z = 3$ if '3s are around the base of the die', or $\{3\}$
  $Z = 4$ if '4s are around the base of the die', or $\{4\}$

- $P\{Z = 1\} = P(\{1\}) = p_Z(1) = \frac{1}{4}$,
  $P\{Z = 2\} = P(\{2\}) = p_Z(2) = \frac{1}{4}$,
  $P\{Z = 3\} = P(\{3\}) = p_Z(3) = \frac{1}{4}$,
  $P\{Z = 4\} = P(\{4\}) = p_Z(4) = \frac{1}{4}$,
  the different ways of writing the probability function emphasises different aspects of

Question 2

Fully define random variables corresponding to the experiments

- ’number of heads shown when two coins are thrown’.
- ’the number of successes out of 4 trials’.
- an urn contains 3 white and 2 blue balls. Two balls are selected without replacement, and £1 is paid out for each blue ball in the selection.

Question 2

Fully define random variables corresponding to the experiments

- 'number of heads shown when two coins are thrown'.

Solution

The easiest sample space is $S = \{(H,H),(H,T),(T,H),(H,H)\}$, as all elements are equally likely.
We have

$$X = 0 \text{ if 'no heads are face up', or, } \{(T,T)\}$$
$$X = 1 \text{ if 'one head is face up', or, } \{(H,T),(T,H)\}$$
$$X = 2 \text{ if 'two heads are face up', or} \{(H,H)\}$$

and the probability function

$$p_X(0) = P\{X = 0\} = P\{TT\} = \frac{1}{4},$$
$$p_X(1) = P\{X = 1\} = P\{TH, HT\} = \frac{1}{2},$$
$$p_X(2) = P\{X = 2\} = P\{HH\} = \frac{1}{4}.$$

- 'the number of successes out of 4 trials'.

If we define $T = \{s, f\}$ then the sample space is given by the cartesian product $S = \{T \times T \times T \times T\}$. $Z =$ 'number of successes out of 4 trials', so

$$Z(z) = \begin{cases} 0 & \text{if 0 successes in event} \\ 1 & \text{if 1 successes} \\ 2 & \text{if 2 successes} \\ 3 & \text{if 3 successes} \\ 4 & \text{if 4 successes} \end{cases}$$

let the probability of success be p and failure $q = 1 - p$

$$p_Z(z) = \begin{cases} \binom{4}{0} q^4 & \text{if Z=0} \\ \binom{4}{1} p q^3 & \text{if Z=1} \\ \binom{4}{2} p^2 q^2 & \text{if Z=2} \\ \binom{4}{3} p^3 q & \text{if Z=3} \\ \binom{4}{4} p^4 & \text{if Z=4} \end{cases}$$

where $\binom{n}{r}$ are binomial coefficients. You may want to write out all 16 possible outcomes explicitly to check you agree with this.

Question 2

Fully define random variables corresponding to the experiments

- an urn contains 3 white and 2 blue balls. Two balls are selected without replacement, and £1 is paid out for each blue ball in the selection.

A sample space is $S = \{(W,W),(W,B),(B,W),(B,B)\}$. We define

$$Y(y) = \begin{cases} 0 & \text{if 0 B, or (W,W)} \\ 1 & \text{if 1 B, or (B,W),(W,B)} \\ 2 & \text{if 2 B, or (B,B)} \end{cases}$$

$$p_Y(y) = \begin{cases} 3/5 \times 2/4 & \text{if Y=0} \\ 3/5 \times 2/4 + 2/5 \times 3/4 & \text{if Y=1} \\ 2/5 \times 1/4 & \text{if Y=2} \end{cases}$$

Question 3

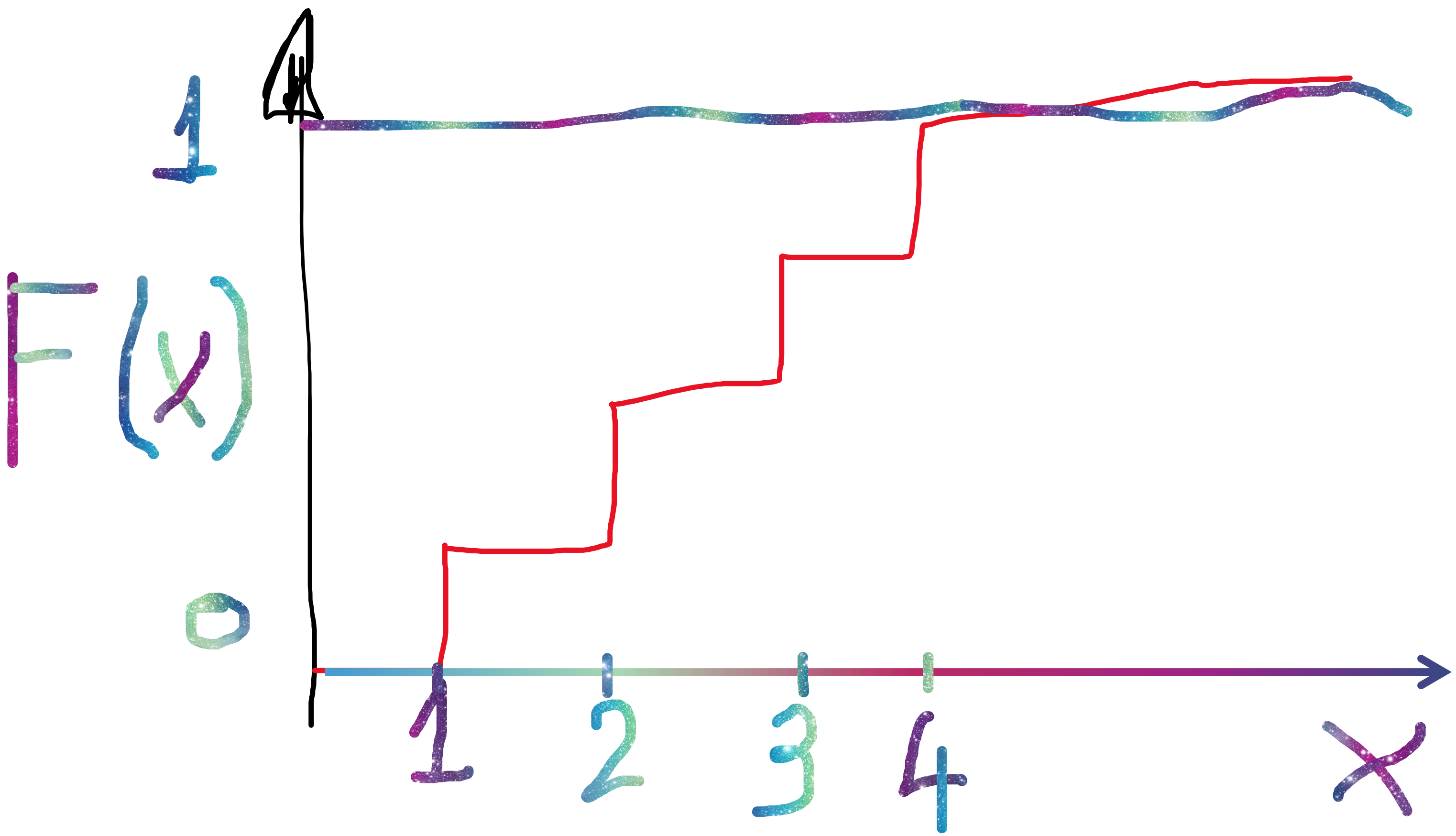
The *cumulative distribution functions (CDF)* of the for the variables in questions 1 and 2 is given by the expression $F(a) = \sum_{\text{all } x \leq a} p_X(x)$. How do the plots of the CDFs look like?

Question 3

The *cumulative distribution functions (CDF)* of the for the variables in questions 1 and 2 is given by the expression $F(a) = \sum_{\text{all } x \leq a} p_X(x)$. How do the plots of the CDFs look like?

Solution

It look like a series of step functions jumping at the values of the random variable.



Question 4

Let X be the discrete random variable *'number of tails shown when two coins are thrown'.*
Define two more random variables:

$X_1 =$ the number of tails shown on the first coin,

$X_2 =$ the number of tails shown on the second coin.

Show that X, X_1 and X_2 are random variables.

Solution

$$p_X(0) = P\{X = 0\} = P\{HH\} = \frac{1}{4}$$

$$p_X(1) = P\{X = 1\} = P\{TH, HT\} = \frac{1}{2}$$

$$p_X(2) = P\{X = 2\} = P\{TT\} = \frac{1}{4}$$

$$p_{X_1}(0) = P\{X_1 = 0\} = P\{HH, HT\} = \frac{1}{2}$$

$$p_{X_1}(1) = P\{X_1 = 1\} = P\{TH, TT\} = \frac{1}{2}$$

$$p_{X_2}(0) = P\{X_2 = 0\} = P\{HH, TH\} = \frac{1}{2}$$

$$p_{X_2}(1) = P\{X_2 = 1\} = P\{HT, TT\} = \frac{1}{2}$$

All have mutually exclusive events associated, and sum of probability mass functions = 1 in each case.

Question 5

In a game 3 dice are rolled. The player bets £1. They get back £1 if they roll a single 5, £2 if 2 fives come up, and £3 if 3 fives come up (and their stake is returned). If no 5s come up they lose their £1 stake.

1. Calculate the value of the probability mass function.
2. Calculate the probability mass function for a game with 4 dice.

Question 5

In a game 3 dice are rolled. The player bets £1. They get back £1 if they roll a single 5, £2 if 2 fives come up, and £3 if 3 fives come up (and their stake is returned). If no 5s come up they lose their £1 stake.

1. Calculate the value of the probability mass function.

1) The sample space of the game is the Cartesian product of rolling three dice:

$$D = \{1, 2, 3, 4, 5, 6\}$$

with $|D| = 6$

$$S = \{D \times D \times D\}$$

and $|S| = 6 \times 6 \times 6 = 216$.
The random variable V was assigned to the values of the possible outcome of the game:

$$V = \begin{cases} -1 & \text{if '0 dice lands with 5 spots on the top face'} \\ 1 & \text{if '1 dice lands with 5 spots on the top face'} \\ 2 & \text{if '2 dice land with 5 spots on the top face'} \\ 3 & \text{if '3 dice land with 5 spots on the top face'} \end{cases}$$

Question 5

In a game 3 dice are rolled. The player bets £1. They get back £1 if they roll a single 5, £2 if 2 fives come up, and £3 if 3 fives come up (and their stake is returned). If no 5s come up they lose their £1 stake.

1. Calculate the value of the probability mass function.

2. Calculate the probability mass function for a game with 4 dice.

for $V = -1$, we have $5 \times 5 \times 5 = 125$ ways to combine with repetition 5 number in group of 3. Therefore $p_V(-1) = \frac{125}{|S|} = \frac{125}{216}$.
For $V = 1$, we have three possible cases:

$$p_V(1) = P\{(5XX)\} + P\{(X5X\} + P\{(XX5\} = \frac{1}{6}\frac{5}{6}\frac{5}{6} + \frac{5}{6}\frac{1}{6}\frac{1}{6} + \frac{5}{6}\frac{5}{6}\frac{1}{6} = \frac{25}{216} + \frac{25}{216} + \frac{25}{216} = \frac{75}{216}$$

where $(X : x_i, i = 1,2,3,4,5,6)$.
For $V = 2$, we have three possible cases:

$$p_V(2) = P\{(55X)\} + P\{(5X5\} + P\{(X55\} = \frac{5}{216} + \frac{5}{216} + \frac{15}{216} = \frac{15}{216}$$

Finally, for $V = 3$, we have one possible case: $p_V(3) = P\{(555)\} = \frac{1}{216}$
2) The previous reasoning can be easily extended to the 4 game dice given the win is based on the number of 5 spot outcomes.

Question 6

The probabilities that three men hit a target are respectively $\frac{1}{6}$, $\frac{1}{4}$, and $\frac{1}{3}$. Each man shoots once at the target.

1. Find the probability p that exactly one of them hits the target,

2. If only one hit the target, what is the probability that it was the first man?

INDEPENDENT EVENTS

Theorem: Two events (E, F) are independent if and only if

$$P(E \cap F) = P(E)P(F)$$

For more than two events things become a bit more restrictive.

Theorem: The events $E_1, E_2, E_3, \cdots E_n$ are said to be mutually independent if for every subset

$E'_1, E'_2, E'_3, \ldots E'_r, r \leq n$

$$P(E'_1 \cap E'_2 \cap E'_3 \cap \cdots \cap E'_r) = P(E'_1) \cdot P(E'_2) \cdot P(E'_3) \cdots P(E'_r)$$

Solution

Consider the events:

A = *{first man hits the target}*,

P = *{second man hits the target}*, and

C = *{third man hits the target}*;

then $P(A) = \frac{1}{6}$,

$P(B) = \frac{1}{4}$, and

$P(C) = \frac{1}{3}$.

The three events are independent, and

$P(\bar{A}) = \frac{5}{6}$,

$P(\bar{B}) = \frac{3}{4}$,

$P(\bar{C}) = \frac{2}{3}$.

Question 6

The probabilities that three men hit a target are respectively $\frac{1}{6}$, $\frac{1}{4}$, and $\frac{1}{3}$. Each man shoots once at the target.

1. Find the probability p that exactly one of them hits the target,

1. Let $E =$ *{exactly one man hits the target}*. Then

$$E = (A \cap \bar{B} \cap \bar{C}) \cup (\bar{A} \cap B \cap \bar{C}) \cup (\bar{A} \cap \bar{B} \cap \bar{C})$$

   In other words, if only one hit the target, then it was either only the first man,

   $A \cap \bar{B} \cap \bar{C}$,

   or only the second man,

   $\bar{A} \cap B \cap \bar{C}$,

   or only the third man,

   $\bar{A} \cap \bar{B} \cap \bar{C}$.

   Since the three events are mutually exclusive, we obtain

$$\begin{aligned} p = P(E) &= P(A \cap \bar{B} \cap \bar{C}) + P(\bar{A} \cap B \cap \bar{C}) + P(\bar{A} \cap \bar{B} \cap \bar{C}) \\ &= P(A)P(\bar{B})P(\bar{C}) + P(\bar{A})P(B)P(\bar{C}) + P(\bar{A})P(\bar{B})P(\bar{C}) \\ &= \frac{1}{6}\frac{3}{4}\frac{2}{3} + \frac{5}{6}\frac{1}{4}\frac{2}{3} + \frac{5}{6}\frac{3}{4}\frac{1}{3} = \frac{1}{12} + \frac{5}{36} + \frac{5}{24} = \frac{31}{72} \end{aligned}$$

Question 6

The probabilities that three men hit a target are respectively $\frac{1}{6}$, $\frac{1}{4}$, and $\frac{1}{3}$. Each man shoots once at the target.

2. If only one hit the target, what is the probability that it was the first man?

2. We seek $P(A|E)$, the probability that the first man hit the target given that only one man hit the target.

   Now

   $A \cap E = A \cap \bar{B} \cap \bar{C}$

   is the event that only the first man hit the target.

   By the result in 1),

   $P(A \cap E) = P(A \cap \bar{B} \cap \bar{C}) = \frac{1}{12}$

   and $P(E) = \frac{31}{72}$;

   hence

$$P(A|E) = \frac{P(A \cap E)}{P(E)} = \frac{\frac{1}{12}}{\frac{31}{72}} = \frac{6}{31}$$

University of Lincoln
School of Mathematics and Physics

MTH1005M PROBABILITY AND STATISTICS PRACTICAL 5

Question 1

Let V be a random variable with probability mass function

$$p_V(v) = \begin{cases} \frac{125}{216} & \text{at } v = -1, \\ \frac{75}{216} & \text{at } v = 1, \\ \frac{15}{216} & \text{at } v = 2, \\ \frac{1}{216} & \text{at } v = 3, \\ 0 & \text{otherwise} \end{cases}$$

calculate the expectation (mean, μ_V), variance (σ_V^2) and standard deviation (σ_V) of V.

- **The expectation value or mean** of a discrete variable, X is given by

$$E[X] = \mu_X = \sum_{x \in R_x} x p(x).$$

and of a continuous random variable Y by

$$E[Y] = \mu_Y = \int_{-\infty}^{\infty} y f_Y(y) dy$$

- **The statistical moments of probability distribution functions** are specific quantitative measure of the shape of a function. The zeroth moment is the total probability (i.e. one, Equation 0.1), the first moment, $E[Y]$, correspond to the mean, and it is given by

  the second central moment of a continuous random variable is given by

$$E[Y^2] = \int_{-\infty}^{\infty} y^2 f_Y(y) dy$$

  the third standardised moment is the skewness, and the fourth standardised moment is the kurtosis.

- The **variance of the random variable** X is defined by

$$\sigma_X^2 = Var(X) = E[(X-\mu)^2]$$

and it can be also written as

$$\sigma_X^2 = Var(X) = E[X^2] - (E[X])^2.$$

Question 1

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calculate the expectation (mean, μ_V), variance (σ_V^2) and standard deviation (σ_V) of V.

SOLUTION 1

- The expectation is $\mathrm{E}[V] = \sum_v v p_V(v) = (-1) \cdot \frac{125}{216} + (1) \cdot \frac{75}{216} + (2) \cdot \frac{15}{216} + (3) \cdot \frac{1}{216} = -\frac{17}{216}$
- The variance is $\sigma_V^2 = \mathrm{E}[V^2] - (\mathrm{E}[V])^2$. We calculate
  $\mathrm{E}[V^2] = \sum_v v^2 p_V(v) = (-1)^2 \cdot \frac{125}{216} + (1)^2 \cdot \frac{75}{216} + (2)^2 \cdot \frac{15}{216} + (3)^2 \cdot \frac{1}{216} = \frac{269}{216}$
  Then
  we get $\sigma_V^2 = \frac{269}{216} - (-\frac{17}{216})^2 = \frac{57815}{216^2} = 1.24$
- The standard deviation σ_V is the square root of the variance. $\sigma_V = \sqrt{\sigma_V^2} = 1.11$

Question 2

Let X be the discrete random variable 'number of tails shown when two coins are thrown'. Define two more random variables X_1 = the number of tails shown on the first coin and X_2 the number of tails shown on the second coin.

- Show that X, X_1 and X_2 are random variables.

- Calculate the values of the expectation and the variance of the random variables.

SOLUTION 2

$$p_X(0) = P\{X = 0\} = P\{HH\} = \frac{1}{4}$$

$$p_X(1) = P\{X = 1\} = P\{TH, HT\} = \frac{1}{2}$$

$$p_X(2) = P\{X = 2\} = P\{TT\} = \frac{1}{4}$$

$$p_{X_1}(0) = P\{X_1 = 0\} = P\{HH, HT\} = \frac{1}{2}$$

$$p_{X_1}(1) = P\{X_1 = 1\} = P\{TH, TT\} = \frac{1}{2}$$

$$p_{X_2}(0) = P\{X_2 = 0\} = P\{HH, TH\} = \frac{1}{2}$$

$$p_{X_2}(1) = P\{X_2 = 1\} = P\{HT, TT\} = \frac{1}{2}$$

All have mutually exclusive events associated, and sum of probability mass functions = 1 in each case.
Lets calculate all the expectations

- $\mathrm{E}[X] = (0) \times \frac{1}{4} + (1) \times \frac{1}{2} + (2) \times \frac{1}{4} = 1$
- $\mathrm{E}[X^2] = (0)^2 \times \frac{1}{4} + (1)^2 \times \frac{1}{2} + (2)^2 \times \frac{1}{4} = \frac{3}{2}$
- $\mathrm{E}[X_1] = (0) \times \frac{1}{2} + (1) \times \frac{1}{2} = \frac{1}{2}$
- $\mathrm{E}[X_1^2] = (0)^2 \times \frac{1}{2} + (1)^2 \times \frac{1}{2} = \frac{1}{2}$
- $\mathrm{E}[X_2] = (0) \times \frac{1}{2} + (1) \times \frac{1}{2} = \frac{1}{2}$
- $\mathrm{E}[X_2^2] = (0)^2 \times \frac{1}{2} + (1)^2 \times \frac{1}{2} = \frac{1}{2}$
- $\mathrm{E}[X_1 X_2] = (0) \cdot (0) \times \frac{1}{4} + (0) \cdot (1) \times \frac{1}{4} + (1) \cdot (0) \times \frac{1}{4} = \frac{1}{2}(1) \cdot (1) \times \frac{1}{4} = \frac{1}{4}$
- $\mathrm{E}[X X_1] = (0) \cdot (0) \times \frac{1}{4} + (1) \cdot (0) \times \frac{1}{4} + (2) \cdot (0) \times \frac{0}{4} + (0) \cdot (1) \times \frac{0}{4} + (1) \cdot (1) \times \frac{1}{4} + (2) \cdot (1) \times \frac{1}{4} = \frac{3}{4}$
- $\sigma_X^2 = \mathrm{E}[X^2] - \left(\mathrm{E}[X]\right)^2 = \frac{3}{2} - (1)^2 = \frac{1}{2}$
- $\sigma_{X_1}^2 = \mathrm{E}[X_1^2] - \left(\mathrm{E}[X_1]\right)^2 = \frac{1}{2} - \frac{1}{2}^2 = \frac{1}{4}$

Review

- **The probability density function** $f_Y(y)$ of a continuous random variable Y is a function whose integral in a given subset $[a,b]$ gives the likelihood of the subset in the sample space of Y:

$$P\{a \leq Y \leq b\} = \int_a^b f_Y(y)\,dy = F_Y(b) - F_Y(a)$$

To be a valid probability density function of a random variable, we must have $f_Y(y) \leq 0$ for all y, and

$$\int_{-\infty}^{\infty} f_Y(y)\,dy = 1 \tag{0.1}$$

- **The cumulative distribution function** for the continuous random variable, Y, is defined as

$$F_Y(a) = P\{Y \leq a\} = \int_{-\infty}^{a} f_Y(y)\,dy.$$

- **The expectation value or mean** of a discrete variable, X is given by

$$E[X] = \mu_X = \sum_{x \in R_x} x p(x).$$

and of a continuous random variable Y by

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- **The statistical moments of probability distribution functions** are specific quantitative measure of the shape of a function. The zeroth moment is the total probability (i.e. one, Equation 0.1), the first moment, $E[Y]$, correspond to the mean, and it is given by

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Question 3

A variable Z has a probability density function

$$f_Z(z) = \begin{cases} Ce^{-0.0001z} & z \geq 0\,, \\ 0 & z < 0 \end{cases}$$

- calculate C to make $f_Z(z)$ a correct probability density function.
- Compute the expectation value of Z

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SOLUTION 4

- $\int_{-\infty}^{\infty} f_Z(z)\mathrm{d}z = 1$ if it is a proper density function. Integrating we get

$$\begin{aligned}
\int_{-\infty}^{\infty} f_Z(z)\mathrm{d}z &= C\left[\frac{e^{-0.0001z}}{-0.0001}\right]\Big|_0^{\infty} \\
&= \frac{C}{-0.0001}\left[(e^{-0.0001\cdot\infty}) - (e^{-0.0001\cdot 0})\right] \\
&= \frac{C}{-0.0001}\cdot(-1)
\end{aligned}$$

$$\implies C = 0.0001$$

We use the fact that $e^{-0.0001\cdot\infty}$, which should really be written $\lim_{L\to\infty} e^{-0.0001\cdot L} = 0$.

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- Compute the expectation value of Z

- $\mathrm{E}[Z] = \int_{-\infty}^{\infty} z \cdot f_Z(z)\mathrm{d}z$ so we just need to do the integral

$$\int_{-\infty}^{\infty} z f_Z(z)\mathrm{d}z = 0.0001 \int_0^{\infty} z e^{-0.0001z}\mathrm{d}z$$

integrating by parts, letting $u = z$ and $dv = 0.001e^{-0.0001z}dz$ we get

$$\mathrm{E}[Z] = -ze^{-0.0001z}\Big|_0^{\infty} - \int_0^{\infty} e^{-0.0001z}dz$$

now the integrated part is equal to zero - when $z = 0$ we multiply by 0, when $z \to \infty$ the exponential drops to zero very quickly, and again there is no contribution. The integral we have already done, it is -1/0.0001 again. And we finally get

$$\mathrm{E}[Z] = 10000$$

Question 4

Calculate the expectation and variance of

$$f_T(t) = \begin{cases} \frac{2}{15} - \frac{2t}{225} & 0 \leq \text{t} \leq 15\,, \\ 0 & \text{otherwise} \end{cases}$$

- **The expectation value or mean** of a discrete variable, X is given by

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and of a continuous random variable Y by

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SOLUTION 5

- the expectation is given by

$$\begin{aligned}
\int_{-\infty}^{\infty} t f_T(t)\mathrm{d}t &= \frac{t^2}{15} - \frac{2t^3}{3\cdot 225}\Big|_{-\infty}^{\infty} \\
&= \frac{t^2}{15} - \frac{2t^3}{3\cdot 225}\Big|_{0}^{15} \\
&= [(\frac{15^2}{15} - \frac{2\cdot 15^3}{3\cdot 225}) - (0)] \\
&= 15 - \frac{2\cdot 15}{3} = 5
\end{aligned}$$

Question 4

Calculate the expectation and variance of

$$f_T(t) = \begin{cases} \frac{2}{15} - \frac{2t}{225} & 0 \leq t \leq 15\,, \\ 0 & \text{otherwise} \end{cases}$$

- the variance is given by $\sigma_T^2 = \mathrm{E}[T^2] - (\mathrm{E}[T])^2$. We calculate

$$\begin{aligned}
\int_{-\infty}^{\infty} t^2 f_T(t)\mathrm{d}t &= \frac{2t^3}{3\cdot 15} - \frac{2t^4}{4\cdot 225}\Big|_{-\infty}^{\infty} \\
&= \frac{2t^3}{3\cdot 15} - \frac{2t^4}{4\cdot 225}\Big|_{0}^{15} \\
&= [(\frac{2\cdot 15^3}{3\dot{1}5} - \frac{2\cdot 15^4}{4\cdot 225}) - (0)] \\
&= \frac{2}{3}15^2 - \frac{\cdot 15^2}{2} = \frac{1}{6}225 = 37.5
\end{aligned}$$

and then subtract 5^2 to get $\sigma_T^2 = 12.5$

Question 5

A variable Y has a probability density function

$$f_Y(y) = \begin{cases} \frac{1}{4}(2y+3) & 0 \leq \mathrm{y} \leq 1\ , \\ 0 & \text{otherwise} \end{cases}$$

- compute the expectation value of Y and aY
- compute the expectation value of Y^2 and $(aY)^2$
- compute the variance of Y and aY
- compute the variance of $(aY + b)$
- would this result apply to other random variables?

Question 5

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- compute the variance of Y and aY
- compute the variance of $(aY+b)$
- would this result apply to other random variables?

Solution 3

- the expectations are given by

$$\begin{aligned}\int_{-\infty}^{\infty} y f_Y(y)\mathrm{d}y &= \frac{2y^3}{3\cdot 4} + \frac{3y^2}{2\cdot 4}\Big|_0^1 \\ &= \frac{2\cdot 1^3}{3\cdot 4} + \frac{3\cdot 1^2}{2\cdot 4} \\ &= \frac{2}{12} + \frac{3}{8} = \frac{13}{24}\end{aligned}$$

$$\int_{-\infty}^{\infty} a y f_Y(t)\mathrm{d}t = a\int_{-\infty}^{\infty} y f_Y(t)\mathrm{d}t = a\mathrm{E}[Y] = a\frac{13}{24}$$

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- compute the expectation value of Y and aY
- compute the expectation value of Y^2 and $(aY)^2$

$$\begin{aligned} \mathrm{E}[Y^2] = \int_{-\infty}^{\infty} y^2 f_Y(y)\mathrm{d}y &= \frac{2y^4}{4\cdot 4} + \frac{3y^3}{3\cdot 4}\Big|_0^1 \\ &= \frac{2\cdot 1^4}{4\cdot 4} + \frac{3\cdot 1^3}{3\cdot 4} \\ &= \frac{2}{16} + \frac{3}{12} = \frac{18}{48} \end{aligned}$$

and

$$\mathrm{E}[(aY)^2] = \int_{-\infty}^{\infty} (ay)^2 f_Y(y)\mathrm{d}y = a^2\mathrm{E}[Y^2] = a^2\frac{18}{48}$$

Question 5

A variable Y has a probability density function

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- compute the expectation value of Y^2 and $(aY)^2$
- compute the variance of Y and aY
- compute the variance of $(aY + b)$
- would this result apply to other random variables?

- the variance of Y is

$$\sigma_Y^2 = \mathrm{E}[Y^2] - (\mathrm{E}[Y])^2 = \frac{18}{48} - \left(\frac{13}{24}\right)^2$$

and the variance of aY is

$$\sigma_{aY}^2 = \mathrm{E}[(aY)^2] - (\mathrm{E}[aY])^2 = a^2\frac{18}{48} - \left(a\frac{13}{24}\right)^2 = a^2\sigma_Y^2$$

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- compute the variance of $(aY+b)$
- would this result apply to other random variables?

- The expectation of $aY+b$ will be $\mathrm{E}[aY+b] = a\mathrm{E}[Y]+b$, the expectation of $\mathrm{E}[(aY+b)^2]$ will be given by

$$\begin{aligned} \mathrm{E}[(aY+b)^2] &= \mathrm{E}[(a^2Y^2+2abY+b^2)] \\ &= a^2\mathrm{E}[Y^2]+2ab\mathrm{E}[Y]+b^2 \end{aligned}$$

so the variance of $aY+b$ will be

$$\begin{aligned} \sigma^2_{aY+b} &= \mathrm{E}[(aY+b)^2]-(\mathrm{E}[aY+b])^2 \\ &= a^2\mathrm{E}[Y^2]+2ab\mathrm{E}[Y]+b^2-(a\mathrm{E}[Y]+b)^2 \\ &= a^2\mathrm{E}[Y^2]+2ab\mathrm{E}[Y]+b^2-(a^2(\mathrm{E}[Y])^2+2ab\mathrm{E}[Y]+b^2) \\ &= a^2\mathrm{E}[Y^2]-a^2(\mathrm{E}[Y])^2 \\ &= a^2\sigma^2_Y \end{aligned}$$

- none of this depended on the form of Y, so it applies to any random variable. But only the linear function $aX+c$.

UNIVERSITY OF LINCOLN
SCHOOL OF MATHEMATICS AND PHYSICS

MTH1005M PROBABILITY AND STATISTICS PRACTICAL 6

- **The expectation value or mean** of a discrete variable, X is given by

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- the variance of Y is

$$\sigma_Y^2 = \mathrm{E}[Y^2] - (\mathrm{E}[Y])^2 = \frac{18}{48} - \left(\frac{13}{24}\right)^2$$

and the variance of aY is

$$\sigma_{aY}^2 = \mathrm{E}[(aY)^2] - (\mathrm{E}[aY])^2 = a^2\frac{18}{48} - \left(a\frac{13}{24}\right)^2 = a^2\sigma_Y^2$$

SUMMARY

Definition

Let $g(X)$ be any function of a random variable X. Then

$$\mathrm{E}[g(X)] = \sum_{x \in R_X} g(x) \cdot p(x)$$

or for the continuous random variable Z

$$\mathrm{E}[g(Z)] = \int_{-\infty}^{\infty} g(z) \cdot f(z) \mathrm{d}z$$

---

if X is a random variable, then

- $\mathrm{E}[a] = a$
- $\mathrm{E}[aX] = a\mathrm{E}[X]$
- $\mathrm{E}[g_1(X) + g_2(X)] = \mathrm{E}[g_1(X)] + \mathrm{E}[g_2(X)]$, where $g_1(X)$ and $g_2(X)$ are any functions of X.

these define the properties of a linear operator.

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$$\begin{aligned} \mathrm{E}[(aY+b)^2] &= \mathrm{E}[(a^2Y^2+2abY+b^2)] \\ &= a^2\mathrm{E}[Y^2]+2ab\mathrm{E}[Y]+b^2 \end{aligned}$$

so the variance of $aY+b$ will be

$$\begin{aligned} \sigma^2_{aY+b} = \mathrm{E}[(aY+b)^2]-(\mathrm{E}[aY+b])^2 \\ &= a^2\mathrm{E}[Y^2]+2ab\mathrm{E}[Y]+b^2-(a\mathrm{E}[Y]+b)^2 \\ &= a^2\mathrm{E}[Y^2]+2ab\mathrm{E}[Y]+b^2-(a^2(\mathrm{E}[Y])^2+2ab\mathrm{E}[Y]+b^2) \\ &= a^2\mathrm{E}[Y^2]-a^2(\mathrm{E}[Y])^2 \\ &= a^2\sigma^2_Y \end{aligned}$$

- none of this depended on the form of Y, so it applies to any random variable. But only the linear function $aX+c$.

USEFUL IDENTITY OF THE VARIANCE

$$\boldsymbol{Var(aX + b) = a^2 Var(X)}$$

a shift of the distribution doesn't change its spread.

Question 2

A service station has both self-service and full-service islands. On each island, there is a single regular unleaded pump with two hoses. Let X denote the number of hoses being used on the self-service island at a particular time, and let Y denote the number of hoses on the full-service island in use at that time. The joint pmf of X and Y appears in the accompanying tabulation.

$p(x,y)$		y = 0	y = 1	y = 2
x	0	0.10	0.04	0.02
	1	0.08	0.20	0.06
	2	0.06	0.14	0.30

a. What is $P(X = 1 \text{ and } Y = 1)$?

b. Compute $P(X \leq 1 \text{ and } Y \leq 1)$.

c. Give a word description of the event $\{X \neq 0 \text{ and } Y \neq 0\}$, and compute the probability of this event.

d. Compute the marginal pmf of X and of Y. Using $p_X(x)$, what is $P(X \leq 1)$?

e. Are X and Y independent random variables? Explain.

$p(x, y)$		y 0	1	2
	0	0.10	0.04	0.02
x	1	0.08	0.20	0.06
	2	0.06	0.14	0.30

a. What is $P(X = 1 \text{ and } Y = 1)$?

b. Compute $P(X \leq 1 \text{ and } Y \leq 1)$.

c. Give a word description of the event $\{X \neq 0 \text{ and } Y \neq 0\}$, and compute the probability of this event.

d. Compute the marginal pmf of X and of Y. Using $p_X(x)$, what is $P(X \leq 1)$?

e. Are X and Y independent random variables? Explain.

SOLUTION 2

a. $P(X = 1, Y = 1) = p(1,1) = .20.$

b. $P(X \leq 1 \text{ and } Y \leq 1) = p(0,0) + p(0,1) + p(1,0) + p(1,1) =$ 0.1+0.04+0.08+0.2=0.42

c. At least one hose is in use at both islands. $P(X \neq 0 \text{ and } Y \neq 0) = p(1,1) + p(1,2) + p(2,1) + p(2,2) = .70.$

d. By summing row probabilities, $p_X(x) = .16, .34, .50$ for $x = 0, 1, 2$, By summing column probabilities, $p_Y(y) = .24, .38, .38$ for $y = 0, 1, 2$. $P(X \leq 1) = p_X(0) + p_X(1) = .50.$

e. $p(0,0) = .10$, but $p_X(0) \cdot p_Y(0) = (.16)(.24) = .0384 \neq .10$, so X and Y are not independent.

Question 3

Ada is a room usage surveyor. She took data on the number of people using classrooms in the MTH building.

	small	medium	large
morning	17	7	3
afternoon	8	19	15

Let X be a random variable taking values 0,1,2 for small, medium and large room sizes.

Also, let Y be a random variable taking values 0 and 1 for when the observation was in the morning or afternoon, respectively.

- Determine the joint probability mass function for X and Y, and the marginal probabilities of X and Y.
- Are X and Y independent? - (for independence $P(X = x, Y = y) = P(X = x)P(Y = y)$ for all x, y.
- Find the covariance and correlation of X and Y.

- Determine the joint probability mass function for X and Y, and the marginal probabilities of X and Y.
- Are X and Y independent? - (for independence $P(X=x, Y=y) = P(X=x)P(Y=y)$ for all x, y.
- Find the covariance and correlation of X and Y.

SOLUTION 3

Total number of observations = 69

	$X=0$	$X=1$	$X=2$	sum
$Y=0$	17/69	7/69	3/69	27/69
$Y=1$	8/69	19/69	15/69	42/69
sum	25/69	26/69	18/69	69/69

The marginal values give $p(x)$ and $p(y)$. Test whether $p_{XY}(0,0) = p_X(0)p_Y(0)$.

From the table we get $17/69 \neq 27/69 \times 25/69$ so not independent. We can stop here as soon as we find an dependent pair of values.

- The **variance of the random variable** X is defined by

$$\sigma_X^2 = Var(X) = E[(X-\mu)^2]$$

  and it can be also written as

$$\sigma_X^2 = Var(X) = E[X^2] - (E[X])^2.$$

- **The expectation of a joint discrete random variable** is defined as

$$E[X,Y] = \sum_y \sum_x xyp(x,y)$$

- **The covariance** of a discrete random variable is defined as

$$\begin{aligned} Cov(X_1, X_2) &= E\left[(X_1 - E[X_1])(X_2 - E[X_2])\right] \\ &= E[X_1 X_2] - E[X_1]E[X_2] \\ &= \sum_{all\,x_1} \sum_{all\,x_2} (x_1 - \mu_{X_1})(x_2 - \mu_{X_2})\, p(x_1, x_2) \end{aligned}$$

- In general it can be shown that a positive Cov(X,Y) is an indication that Y increases when X does. A negative Cov(X,Y) is an indication that Y decreases when X increases. A better indicator than the covariance is **the correlation** that is defined as

$$Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$$

It can be shown that the correlation will always have a value betwen -1 and $+1$. The significance of the correlation is similar to just discussed for the covariance:

  - Positive correlation between two variables means that X increases as Y increases.
  - Negative correlation means that X decreases as Y increases.

We need to calculate $\mathrm{E}[X], \mathrm{E}[Y], \mathrm{E}[XY]$ and the corresponding variances:

$$\mathrm{E}[X] = (0)\cdot 25/69 + (1)\cdot 26/69 + (2)\cdot 18/69 = 62/69$$

$$\mathrm{E}[X^2] = (0)\cdot 25/69 + (1)^2\cdot 26/69 + (2)^2\cdot 18/69 = 98/69$$

$$\mathrm{E}[Y] = (0)\cdot 27/69 + (1)\cdot 42/69 = 42/69$$

$$\mathrm{E}[Y^2] = (0)\cdot 27/69 + (1)^2\cdot 42/69 = 42/69$$

$$\mathrm{E}[XY] = (0)\cdot(0)\cdot(17/69) + (1)\cdot(0)\cdot(7/69) + (2)\cdot(0)\cdot(3/69) + (0)\cdot(1)\cdot(8/69) + (1)\cdot(1)\cdot(19/69) + (2)\cdot(1)\cdot(15/69) = 49/69$$

$$\sigma_X^2 = \mathrm{E}[X^2] - \left(\mathrm{E}[X]\right)^2 = 98/69 - (62/69)^2 = 0.62$$

$$\sigma_Y^2 = \mathrm{E}[Y^2] - \left(\mathrm{E}[Y]\right)^2 = 42/69 - (42/69)^2 = 0.23$$

$$\mathrm{cov}(X,Y) = 49/69 - (62/69)(42/69) = 0.17$$

$$\mathrm{corr}(X,Y) = 0.17/\sqrt{0.62\times 0.23} = 0.45$$

so moderate positive correlation value.

Question 4

Let X be the discrete random variable ’number of tails shown when two coins are thrown’.

Define two more random variables X_1 = the number of tails shown on the first coin and X_2 the number of tails shown on the second coin.

- Calculate the covariances and correlations of the random variables.

SOLUTION 2

$$p_X(0) = P\{X = 0\} = P\{HH\} = \frac{1}{4}$$

$$p_X(1) = P\{X = 1\} = P\{TH, HT\} = \frac{1}{2}$$

$$p_X(2) = P\{X = 2\} = P\{TT\} = \frac{1}{4}$$

$$p_{X_1}(0) = P\{X_1 = 0\} = P\{HH, HT\} = \frac{1}{2}$$

$$p_{X_1}(1) = P\{X_1 = 1\} = P\{TH, TT\} = \frac{1}{2}$$

$$p_{X_2}(0) = P\{X_2 = 0\} = P\{HH, TH\} = \frac{1}{2}$$

$$p_{X_2}(1) = P\{X_2 = 1\} = P\{HT, TT\} = \frac{1}{2}$$

$$p(X_1 = 0, X_2 = 0) = P\{HH\} = \frac{1}{4}$$
$$p(X_1 = 1, X_2 = 0) = P\{TH\} = \frac{1}{4}$$
$$p(X_1 = 0, X_2 = 1) = P\{HT\} = \frac{1}{4}$$
$$p(X_1 = 1, X_2 = 1) = P\{TT\} = \frac{1}{4}$$

$$p(X = 0, X_1 = 0) = P\{HH\} = \frac{1}{4}$$
$$p(X = 1, X_1 = 0) = P\{TH\} = \frac{1}{4}$$
$$p(X = 2, X_1 = 0) = P\{\} = 0$$
$$p(X = 0, X_1 = 1) = P\{\} = 0$$
$$p(X = 1, X_1 = 1) = P\{TH\} = \frac{1}{4}$$
$$p(X = 2, X_1 = 1) = P\{TT\} = \frac{1}{4}$$

All have mutually exclusive events associated, and sum of probability mass functions = 1 in each case.
Lets calculate all the expectations

- $\mathrm{E}[X] = (0) \times \frac{1}{4} + (1) \times \frac{1}{2} + (2) \times \frac{1}{4} = 1$
- $\mathrm{E}[X^2] = (0)^2 \times \frac{1}{4} + (1)^2 \times \frac{1}{2} + (2)^2 \times \frac{1}{4} = \frac{3}{2}$
- $\mathrm{E}[X_1] = (0) \times \frac{1}{2} + (1) \times \frac{1}{2} = \frac{1}{2}$
- $\mathrm{E}[X_1^2] = (0)^2 \times \frac{1}{2} + (1)^2 \times \frac{1}{2} = \frac{1}{2}$
- $\mathrm{E}[X_2] = (0) \times \frac{1}{2} + (1) \times \frac{1}{2} = \frac{1}{2}$
- $\mathrm{E}[X_2^2] = (0)^2 \times \frac{1}{2} + (1)^2 \times \frac{1}{2} = \frac{1}{2}$
- $\mathrm{E}[X_1 X_2] = (0) \cdot (0) \times \frac{1}{4} + (0) \cdot (1) \times \frac{1}{4} + (1) \cdot (0) \times \frac{1}{4} = \frac{1}{2}(1) \cdot (1) \times \frac{1}{4} = \frac{1}{4}$
- $\mathrm{E}[X X_1] = (0) \cdot (0) \times \frac{1}{4} + (1) \cdot (0) \times \frac{1}{4} + (2) \cdot (0) \times \frac{0}{4} + (0) \cdot (1) \times \frac{0}{4} + (1) \cdot (1) \times \frac{1}{4} + (2) \cdot (1) \times \frac{1}{4} = \frac{3}{4}$
- $\sigma_X^2 = \mathrm{E}[X^2] - \left(\mathrm{E}[X]\right)^2 = \frac{3}{2} - (1)^2 = \frac{1}{2}$
- $\sigma_{X_1}^2 = \mathrm{E}[X_1^2] - \left(\mathrm{E}[X_1]\right)^2 = \frac{1}{2} - \frac{1}{2}^2 = \frac{1}{4}$

- **The covariance** of a discrete random variable is defined as

$$\begin{aligned} Cov(X_1, X_2) &= E\left[(X_1 - E[X_1])(X_2 - E[X_2])\right] \\ &= E[X_1 X_2] - E[X_1]E[X_2] \\ &= \sum_{all\,x_1} \sum_{all\,x_2} (x_1 - \mu_{X_1})(x_2 - \mu_{X_2})\, p(x_1, x_2) \end{aligned}$$

- In general it can be shown that a positive Cov(X,Y) is an indication that Y increases when X does. A negative Cov(X,Y) is an indication that Y decreases when X increases. A better indicator than the covariance is **the correlation** that is defined as

$$Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$$

  It can be shown that the correlation will always have a value betwen -1 and $+1$. The significance of the correlation is similar to just discussed for the covariance:

  - Positive correlation between two variables means that X increases as Y increases.
  - Negative correlation means that X decreases as Y increases.

and now the covariances and correlations

$$\mathrm{cov}(X_1, X_2) = \mathrm{E}[X_1 X_2] - \mathrm{E}[X_1]\mathrm{E}[X_2] = \frac{1}{4} - \frac{1}{2} \cdot \frac{1}{2} = 0$$

$$\mathrm{cov}(X, X_1) = \mathrm{E}[X X_1] - \mathrm{E}[X]\mathrm{E}[X_1] = \frac{3}{4} - 1 \cdot \frac{1}{2} = \frac{1}{4}$$

$$\mathrm{corr}(X, X_1) = \frac{1}{4} / \sqrt{\frac{1}{2} \cdot \frac{1}{4}} = \frac{1}{\sqrt{2}}$$

so there is a strong correlation between X and X_1, as you'd expect from their definitions.

Question 5

Suppose the joint pdf of (X, Y) is given by

$$f(x,y) = \begin{cases} \frac{6}{5}(x+y^2) & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

- Verify that this is a legitimate pdf.
- The probability $P(0 \le x \le \frac{1}{4}, 0 \le y \le \frac{1}{4})$.

The joint probability density function $f_{XY}(x, y)$ of a continuous random variables (X, Y) is a function whose integral in the set of possible values for (X, Y) is the rectangle $D = (x, y) : a \leq x \leq b, u \leq y \leq w.$ gives the likelihood of the subset in the sample space of (X, Y):

$$P\{a \leq X \leq b; u \leq Y \leq w\} = \int_a^b \int_u^w f_{XY}(x, y) dx dy$$

To be a valid probability density function of a random variable, we must have

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y) dx dy = 1 \tag{0.1}$$

SOLUTIONS

$$f(x, y) = \begin{cases} \dfrac{6}{5}(x + y^2) & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

To verify that this is a legitimate pdf, note that $f(x, y) \geq 0$ and

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dx\, dy = \int_0^1 \int_0^1 \frac{6}{5}(x + y^2)\, dx\, dy$$

$$= \int_0^1 \int_0^1 \frac{6}{5} x\, dx\, dy + \int_0^1 \int_0^1 \frac{6}{5} y^2\, dx\, dy$$

$$= \int_0^1 \frac{6}{5} x \, dx + \int_0^1 \frac{6}{5} y^2 \, dy = \frac{6}{10} + \frac{6}{15} = 1$$

$$= \frac{6}{10} + \frac{6}{15}$$

$$= 1$$

The probability that neither facility is busy more than one-quarter of the time is

$$P\left(0 \le X \le \frac{1}{4}, 0 \le Y \le \frac{1}{4}\right) = \int_0^{1/4} \int_0^{1/4} \frac{6}{5} (x + y^2) \, dx \, dy$$

$$= \frac{6}{5} \int_0^{1/4} \int_0^{1/4} x \, dx \, dy + \frac{6}{5} \int_0^{1/4} \int_0^{1/4} y^2 \, dx \, dy$$

$$= \frac{6}{20} \cdot \frac{x^2}{2}\Bigg|_{x=0}^{x=1/4} + \frac{6}{20} \cdot \frac{y^3}{3}\Bigg|_{y=0}^{y=1/4}$$

$$= \frac{7}{640}$$

$$= .0109$$

UNIVERSITY OF LINCOLN
SCHOOL OF MATHEMATICS AND PHYSICS

MTH1005 PROBABILITY AND STATISTICS
PRACTICAL 7

Question 1

For a discrete set of data [2, 26, 6, 22, 8, 10, 12, 23, 17, 30, 11, 19, 29, 17, 22, 19, 12, 18, 30, 34] find the mean, the median and the mode.

- **Mean.** For a discrete data set $\{x_i \in X, i = 1, \ldots N\}$, the arithmetic mean, also called the mathematical expectation or average, is the central value of the numbers (x_i) in the the set: specifically, the sum of the values divided by the number of values (N)

- **Median.** The median is the value separating the higher half from the lower half of a data sample. For a discrete data set, it correspond to the "middle" value for an set containing a odd number of data or to the average of the two "middle value for a set with an even number of data. The basic advantage of the median in describing data compared to the mean is that it is not skewed so much by extremely large or small values, and so it may give a better idea of a "typical" value.

- **Mode.** The mode of a set of data values is the value that appears most often. It is the value x at which its probability mass function takes its maximum value. In other words, it is the value that is most likely to be sampled.

SOLUTION 2

Mean = 18.35.
Median = 18.5, mean of 18 and 19
Modes are 12, 17, 19, 22, 30.
Using Python, you can obtain the results using the following script

```python
import numpy as np
x = np.array([ 2, 26,  6, 22,  8, 10, 12, 23, 17, 30, 11, 19,
 29, 17, 22, 19, 12, 18, 30
x.sort()

print(x)
print("mean of x is {:.4}".format(x.mean()))
```

```
[ 2  6  8 10 11 12 12 17 17 18 19 19 22 22 23 26 29 30 30 34]
mean of x is 18.35
```

Question 2

A random variable T has probability density function

$$f_T(t) = \begin{cases} 0 & t < -50 \\ \frac{1}{100} & -50 \le t \le 50 \\ 0 & t > 50 \end{cases}$$

- calculate the mean and median of T,
- the standard deviation and interquartile distance of T.

- **Quantiles.** They are cut points dividing the range of a probability distribution into continuous intervals with equal probabilities, or dividing the observations in a sample in the same way. The 4-quantiles are called **quartiles, Q**. The difference between upper (Q_3) and lower (Q_1) quartiles is also called the interquartile range, $\mathbf{IQR = Q_3 - Q_1}$ The n-quartile of a continuous variable X can be calculated from the cumulative function as

$$F_Y(Q_n) = \int_0^{Q_n} f(x)dx = \frac{n}{4}$$

- The **variance of the random variable X** is defined by

$$\sigma_X^2 = Var(X) = E[(X - \mu)^2]$$

and it can be also written as

$$\sigma_X^2 = Var(X) = E[X^2] - (E[X])^2.$$

with the second central moment of a continuous random variable given by

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x) dx$$

It is the expected squared distance of a value from the centre of the distribution.

SOLUTION

- the mean and median are both 0 (even function)
- the variance of T is

$$f_T(t) = \begin{cases} 0, & t < -50 \\ 1/100, & -50 \leq t \leq 50 \\ 0, & t > 50 \end{cases}$$

$$\sigma_T^2 = \int_{-50}^{50} t^2/100 \, dt - 0^2 = \left[t^3/300 \right]_{-50}^{50} = 250000/300$$

so $\sigma_T = 28.87$.
To find the quartiles we need the cumulative distribution function

$$F_T(t) = \begin{cases} 0, & t < -50 \\ t/100, & -50 \leq t \leq 50 \\ 1, & t > 50 \end{cases}$$

so $Q_1 = -25, Q_3 = 25$ and the interquartile distance = 50.

Question 3

A random variable Z has a probability density function

$$f_Z(z) = \begin{cases} 0.1e^{-0.1z} & z \geq 0\,, \\ 0 & z < 0 \end{cases}$$

- find the mean of Z
- find the median of Z
- find the interquartile distance of Z

SOLUTION

- First we need the expectation of Z

$$\mathrm{E}[Z] = \int_0^\infty 0.1 z e^{-0.1z} dz$$

integrating by parts, letting $u = z$ and $dv = 0.1e^{-0.1z}dz$ we get

$$\mathrm{E}[Z] = -ze^{-0.1z}\Big|_0^\infty - \int_0^\infty e^{-0.1z} dz$$

the integrated part, $-ze^{-0.1z}\Big|_0^\infty$, goes to zero at both limits. And we are left with

$$\begin{aligned}\mathrm{E}[Z] &= -\int_0^\infty e^{-0.1z} dz \\ &= -\left[10e^{-0.1z}\right]_0^\infty \\ &= -\left[0 - 10\right] \\ &= 10\end{aligned}$$

- to find the median we need to integrate $f_Z(z)$. We get

$$
\begin{aligned}
F_Z(a) &= \int_0^a f(z)dz \\
&= \left[\frac{0.1}{-0.1}e^{-0.1z}\right]_0^a \\
&= (1 - e^{-0.1a})
\end{aligned}
$$

and then for median (2nd quartile, Q_2), first quartile, Q_1, second quartile, Q_3, we set it equal to 0.5, 0.25 or 0.75. We get

$$Q_n = -10\ln(1 - n/4)$$

and get $Q_1 = 2.87, Q_2 = 6.93, Q_3 = 13.86$.

Question 4

Fit a least-squares line to the data in the following Table:

x	1	3	4	6	8	9	11	14
y	1	2	4	4	5	7	8	9

using

- x as independent variable,
- x as dependent variable.

SOLUTIONS

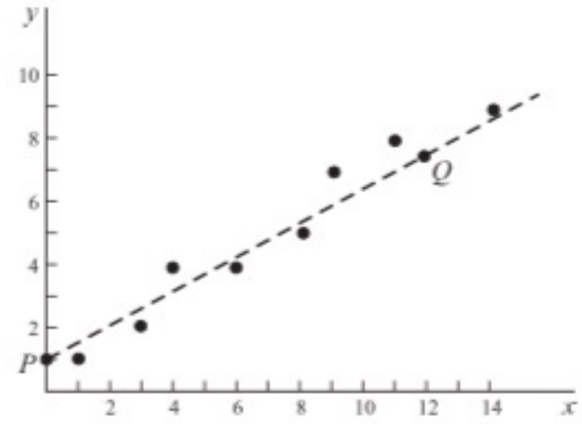
(a) The equation of the line is $y = a + bx$. The normal equations are

$$\sum y = an + b\sum x$$
$$\sum xy = a\sum x + b\sum x^2$$

x	y	x^2	xy	y^2
1	1	1	1	1
3	2	9	6	4
4	4	16	16	16
6	4	36	24	16
8	5	64	40	25
9	7	81	63	49
11	8	121	88	64
14	9	196	126	81
$\sum x = 56$	$\sum y = 40$	$\sum x^2 = 524$	$\sum xy = 364$	$\sum y^2 = 256$

$$a = \frac{\left(\sum y\right)\left(\sum x^2\right) - \left(\sum x\right)\left(\sum xy\right)}{n\sum x^2 - \left(\sum x\right)^2} = \frac{(40)(524) - (56)(364)}{(8)(524) - (56)^2} = \frac{6}{11} \quad \text{or} \quad 0.545$$

$$b = \frac{n\sum xy - \left(\sum x\right)\left(\sum y\right)}{n\sum x^2 - \left(\sum x\right)^2} = \frac{(8)(364) - (56)(40)}{(8)(524) - (56)^2} = \frac{7}{11} \quad \text{or} \quad 0.636$$



(b) If x is considered as the dependent variable and y as the independent variable, the equation of the least-squares line is $x = c + dy$ and the normal equations are

$$\sum x = cn + d\sum y$$
$$\sum xy = c\sum y + d\sum y^2$$

$$c = \frac{\left(\sum x\right)\left(\sum y^2\right) - \left(\sum y\right)\left(\sum xy\right)}{n\sum y^2 - \left(\sum y\right)^2} = \frac{(56)(256) - (40)(364)}{(8)(256) - (40)^2} = -0.50$$

$$d = \frac{n\sum xy - \left(\sum x\right)\left(\sum y\right)}{n\sum y^2 - \left(\sum y\right)^2} = \frac{(8)(364) - (56)(40)}{(8)(256) - (40)^2} = 1.50$$

Question 5

A random variable T has a probability density function

$$f_T(t) = \begin{cases} \frac{2}{15} - \frac{2t}{225} & 0 \leq \mathrm{t} \leq 15\,, \\ 0 & \text{otherwise} \end{cases}$$

- calculate the mean and median of T
- the standard deviation and interquartile distance of T

SOLUTION

The mean is

$$E[T] = \int_0^{15} t \cdot f_T(t)\,dt = \left[\frac{2t^2}{2\cdot 15} - \frac{2t^3}{3\cdot 225}\right]_0^{15} = 15 - 10 = 5$$

To find the median we should calculate the cumulative distribution function $F(t)$ and set it equal to 0.5.

$$\begin{aligned} F(a) &= \int_{-\infty}^{a} f(t)\,dt = \int_0^a f(t)\,dt \\ &= \left[\frac{2t}{15} - \frac{t^2}{225}\right]_0^a \\ &= \frac{2a}{15} - \frac{a^2}{225} \\ &= 0.5 \end{aligned}$$

Which gives a quadratic for a:

$$\frac{2a}{15} - \frac{a^2}{225} - 0.5 = 0 \implies a^2 - 30a + 225/2 = 0$$

so

$$\begin{aligned} a &= \frac{30 \pm \sqrt{(}30^2 - 4 \times 225/2)}{2} \\ &= \frac{30 \pm \sqrt{(}450)}{2} \\ &= \frac{30 \pm 21.21}{2} \end{aligned}$$

only the negative root makes sense, as we know that there is only a distribution of t values between 0 and 15, so median = 4.39

- the standard deviation needs us to calculate

$$
\begin{aligned}
\mathrm{E}[T^2] &= \int_0^{15} t^2\left(\frac{2}{15} - \frac{2t}{225}\right) dt \\
&= \left[\frac{2t^3}{45} - \frac{2t^4}{4 \cdot 225}\right]_0^{15} \\
&= 150 - 225/2 = 37.5
\end{aligned}
$$

then $\sigma_T = \sqrt{\mathrm{E}[T^2] - \mathrm{E}[T]^2} = \sqrt{37.5 - 5^2} = \sqrt{(}12.5)$.
To find the interquartile distance, we need the quartiles, found from solutions of

$$Q_1^2 - 30Q_1 + 225 \times \frac{1}{4} = 0$$

and

$$Q_3^2 - 30Q_3 + 225 \times \frac{3}{4} = 0$$

giving $Q_1 = (30 - \sqrt{675})/2 = 2.01$ and $Q_3 = (30 - \sqrt{225})/2 = 7.5$.
So the interquartile distance ≈ 5.5.

Question 6

Given $\mathrm{E}[T] = 5, \sigma_T^2 = 12.5$, use the Chebyshev inequality

$$P\left(|T - \mathrm{E}[T]| \geq k\right) \leq \frac{\sigma_T^2}{k^2}$$

to estimate the probability that - $2.5 \leq T \leq 7.5$ - $0 \leq T \leq 10$, compare the obtained values with the real probability of 0.44 and 0.89, respectivelly.

- **The Chebyshev inequality.** This powerful relation guarantees that, for a wide class of probability distributions, no more than a certain fraction of values can be more than a certain distance from the mean. It is defined as

$$P\left(|X - \mathrm{E}[X]| \geq k\right) \leq \frac{\sigma_X^2}{k^2}$$

where $E[X]$ is the expectation value, σ_X^2 is the variance of the random variable X. The relation tell us that no more than $1/k^2$ of the distribution's values can be more than k standard deviations away from the mean.

Solution 1

- $\mathrm{E}[X] = 5, \sigma_X^2 = 12.5$. So we are looking at probability that X is greater than $k = 2.5$ away from the mean: $P\big(|X - \mathrm{E}[T]| \geq 2.5\big) \leq \frac{12.5}{2.5^2} = 2.0$
- $\mathrm{E}[X] = 5, \sigma_X^2 = 12.5$. So we are looking at probability that X is greater than $k = 5$ away from the mean: $P\big(|X - \mathrm{E}[T]| \geq 5\big) \leq \frac{12.5}{5^2} = 0.5$

These values are consistent with the correct ones, but in the first case, the Chebyshev inequality is useless.

UNIVERSITY OF LINCOLN
SCHOOL OF MATHEMATICS AND PHYSICS

MTH1005 PROBABILITY AND STATISTICS
PRACTICAL 8

QUESTION 1

Buses arrive at a bus stop at 15-minute intervals starting at 7am(!). The buses are always perfectly on time, so arrive at 7:00, 7:15, 7:30, 7:45 ...

If a passenger arrives at the stop at a time that is uniformly distributed between 7 and 7:30, find the probability that she waits

- less than 5 minutes for a bus;
- at least 12 minutes for a bus.

PROBABILITY DISTRIBUTIONS

- The probability density function for a **uniform distribution** is defined as

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

and the probability between $\alpha < X < \beta$ is given by

$$P(\alpha < X < \beta) = \int_{\alpha}^{\beta} f_X(x)\,dx = \frac{\beta - \alpha}{b - a}.$$

Solution 2

Let X denote the time in minutes past 07:00 that the passenger arrives at the stop.
X is uniformly distributed over $(0,30)$ so has probability density function

$$f_X = 1/30, 0 \le X \le 30$$

and the probability

$$P(a < X < b) = \int_a^b f_X(x)dx = \frac{b-a}{30}$$

- the passenger will wait less than 5 minutes if she arrives between 7:10 and 7:15, or between 7:25 and 7:30.

$$P(< 5 \text{ minute wait}) = P(10 < X < 15) + P(25 < X < 30) = 5/30 + 5/30 = 1/3$$

- the passenger will wait more than 12 minutes if she arrives between 7:00 and 7:03, or between 7:15 and 7:18.

$$P(> 12 \text{ minute wait}) = P(0 < X < 3) + P(15 < X < 18) = 3/30 + 3/30 = 1/5$$

QUESTION 2

A machine producing thin metal strips is estimated to average one minor flaw in 24 metres. If flaws occur randomly, which type of probability distribution would be appropriate to model the system?

And, what is the probability of:

- 2 flaws in a 3 metre length?
- no flaws in a 12 metre length?

- The probability density function for a **Poisson distribution** is defined as

$$f_X(x) = \frac{e^{-\mu}\mu^x}{x!}$$

The expectation is given by

$$E[X] = \mu,$$

and the variance by

$$\sigma_X^2 = \mu.$$

SOLUTION 2

Let the random variable X represent the number of flaws.
The events seem to be independent and random, so if we expected 1 flaw in 24 metres, we expect 1/8 flaws in 3 metres.
Therefore we think X should be Poisson distributed with $\mu = 0.125$. So

$$p(2) = e^{-0.125}\frac{0.125^2}{2!} = 0.0069$$

We'd expect 0.5 flaws in a 12 metre length so

$$p(0) = e^{-0.5}\frac{0.5^0}{0!} = 0.6065$$

QUESTION 3

Equivalent probabilities. Show that (or convince yourself that):

- $P(|X| < 0.5) = P(-0.5 < X < 0.5)$
- $P(-0.5 \leq X \leq 0.5) = P(-1 \leq 2X \leq 1)$
- $P\left(|2X - 6| < 1.0\right) = P(-3.5 < X < 3.5)$

And if μ is a real number and σ is a positive real number:

- $P(|X| \leq k) = P(\frac{-k-\mu}{\sigma} \leq \frac{X-\mu}{\sigma} \leq \frac{k-\mu}{\sigma})$
- $P(|X| > k) = 1 - P(\frac{-k-\mu}{\sigma} \leq \frac{X-\mu}{\sigma} \leq \frac{k-\mu}{\sigma})$

Finally

- $P(X^2 \leq 2) = P(-\sqrt{2} \leq X \leq \sqrt{2})$
- $P(X^2 > 2) = 1 - P(-\sqrt{2} \leq X \leq \sqrt{2})$

SOLUTION 3

The main point to realise is that the fact that these inequalities are inside the $P(\ldots)$ does not matter - we are working with inequalities of standard real numbers using rules that should be familiar to you. If the inequalities define equivalent parts of the real line, the probabilities of these events must be the same.

- $P(|X| < 0.5)$ is the same as $P(\{0 \le X \le 0.5\} \cup \{-0.5 \le X \le 0.0)\} = P(-0.5 < X < 0.5)$
- we can multiply all terms by 2, a positive number without effecting the order of the statements, so
- $P(-0.5 \le X \le 0.5) = P(-2 \times 0.5 \le 2 \times X \le 2 \times 0.5) = P(-1 \le 2X \le 1)$
- $P\left(|2X - 6| < 1.0\right) = P(-1 < 2X - 6 < 1) = P(-7 < 2X < 7) = P(-3.5 < X < 3.5)$
- $P(|X| \le k) = P(-k \le X \le k) = P(-k - \mu \le X - \mu \le k - \mu) = P(\frac{-k-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{k-\mu}{\sigma})$
- $P(|X| > k) = 1 - P(|X| \le k)$, then use the previous subquestion
- $P(X^2 \le 2) = P(-\sqrt{2} \le X \le \sqrt{2})$
- $P(X^2 > 2) = 1 - P(X^2 \le 2)$, then use the previous subquestion

QUESTION 4

Tomato computers are known to be 0.3% defective. If a sample of 350 is taken randomly, calculate the probability of observing more than 2 defective computers in the sample. Do this using

- binomial distribution
- poisson distribution

how do they compare?

- The probability density function for a **binomial distribution** is defined as

$$f_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

with $x = 0, 1, \ldots, n$

The expectation is given by

$$E[X] = np$$

and the variance by

$$\sigma_X^2 = E[X^2] - E[X]^2 = np(1-p)$$

SOLUTION 4

This is a binomial distributed case with $n = 350$ (each member is sampled independently) with a probability of 'succcess', p, of 0.003.
Since n is large (>50 ish) and p is small (< 0.1), it could also be approximated as a Poisson distribution with mean np.

- We use a binomial distribution with $n = 350$ and $p = 0.003$. $P(X > 2) = 1 - p(0) - p(1) - p(2)$, where $p(x) = \binom{n}{x} p^x (1-p)^{n-x}$.
  Using a fancy calculator $p(0) = 0.349$, $p(1) = 0.368$ and $p(2) = 0.193$ to get 0.089 as the probability of getting more than two defective 'tomatos'.

- Using the Poisson approximation we have $\mu = np = 1.05$ and $p(x) = e^{-1.05} \frac{(1.05)^x}{x!}$ and using calculator or similar $p(0) = 0.350$, $p(1) = 0.367$, $p(2) = 0.193$. So $P(X > 2) = 1 - 0.350 - 0.367 - 0.193 = 0.090$

- the approximation is almost perfect to our level of precision.

QUESTION 5

Show that the random variable X with a geometric distribution

$$P(X = x) = (1 - p)^x p$$

is a valid random variable if $0 \leq p \leq 1$ and X can take values $0, 1, 2\ldots$
You might need to use the properties of a geometric series:

$$a + ar + ar^2 + ar^3 + ar^4 + \cdots = \sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}, \text{ for } |r| < 1.$$

Review

PROBABILITY DISTRIBUTIONS

The probability density function for a **geometric distribution** is defined as

$$f_X(x) = (1-p)^x p$$

for $0 \leq p \leq 1$.
The expectation is given by

$$E[X] = \frac{1-p}{p}$$

and the variance by

$$\sigma_X^2 = \frac{1-p}{p^2}$$

Solution 5

We need to show that the sum over all values of the probability mass function is 1.

$$\begin{aligned}
\sum_{\text{all } x} P(X = x) &= \sum_{\text{all } x} (1-p)^x p \\
&= p \sum_{\text{all } x} (1-p)^x \\
&= \frac{p}{1-(1-p)} \\
&= 1
\end{aligned}$$

where we used the fact that the sum was a geometric series to go from the 2nd to 3rd lines. Also the condition on p is the same as for the convergence of the geometric series.

Question 6

In the summer of the 1905, the eminent British statistician Karl Pearson (1857-1936) wrote a letter to the Journal *Nature* in which he asked for help on the following mathematical problem:

A man starts from a point O and walks l yards in a straight line; he then turns through any angle whatever and walks another l yards in a second straight line. He repeats this process n times. I require the probability that after these n stretches he is at a distance between r and r + dr from his starting point, O.

The brilliant mathematician and experimental physicist Lord Rayleigh (aka John William Strutt) promptly replayed giving as correct answer the following probability density function:

$$f_R(r) = \begin{cases} 0, & r < 0 \\ \frac{2}{n} e^{-r^2/n}, & r \geq 0 \end{cases}$$

that is now famous as *Rayleigh's pdf.*

Show that if the random variable $\mathbf{X}$ has this pdf then the random variable $\mathbf{Y} = \mathbf{X}^2$ does in fact an exponential pdf.

Karl Pearson,FRS
1857-1857
(source: Wikipedia)

John William Strutt,
3rd Baron Rayleigh
OM,PC,PRS
(1842-1919)

The probability density function for a **exponential distribution** is defined as

$$f_X(x) = \begin{cases} \beta e^{-\beta x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

The expectation is given by

$$E[X] = \frac{1}{\beta}$$

and the variance by

$$\sigma_X^2 = \frac{1}{\beta^2}$$

SOLUTION 6

The pdf is

$$f_X(x) = \begin{cases} 0, & x < 0 \\ \frac{2}{n} e^{-x^2/n}, & x \geq 0 \end{cases}$$

that is, **X** is a Rayleigh random variable. To find $\mathbf{Y} = \mathbf{X}^2$, we first find **Y** distribution function and the differentiate. So,

$$F_Y(y) = P(Y \leq y) = P(X^2 \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y})$$

or, since **X** is never negative

$$F_Y(y) = P(0 \leq X \leq \sqrt{y}) = \int_0^{\sqrt{y}} f_X(x)\,dx = \int_0^{\sqrt{y}} \frac{2}{n} e^{-x^2/n}\,dx$$

This integral is actually doable, but it is not necessary to solve it as we are going to differentiate the result. So,

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{2}{n} e^{-(\sqrt{y})^2/n} \frac{1}{2} \frac{\sqrt{y}}{\sqrt{y}} = \frac{1}{n} e^{-y/n}.$$

That is,

$$f_Y(y) = \begin{cases} 0, & y < 0 \\ \frac{1}{n} e^{-y/n}, & y \geq 0 \end{cases}$$

which is, indeed, the pdf of an exponential random variable, as was to be shown.

Question 7

The lab session simulation of this week consisted in calculating the chance to find the Blackbeard's buried treasure in one of 10 islands. This problem is equivalent to the change to find a coin is in one of n boxes.

The probability that it is in the ith box is p_i. If you search in the ith box and it is there, you find it with probability a_i. Show that the probability p that the coin is in the jth box, given that you have looked in the ith box and not found it is,

$$p = \begin{cases} \frac{p_j}{(1-a_i p_i)}, & \text{if } j \neq i, \\ \frac{(1-a_i)p_i}{(1-a_i p_i)}, & \text{if } j = i \end{cases}$$

SOLUTION 7

We'll call the event that box i was searched and no coin was found E. And the event j that the coin is in box j.
Let p be the conditional probability that the coin is in a particular box - given that box i has been searched and no coin found.
We can express conditional probabilities using Bayes' formula:

$$
\begin{aligned}
p &= \frac{P(j \cap E)}{P(E)} && (0.1)\\
&= \frac{P(E \mid j)P(j)}{\sum_{k=1}^{n} P(E \mid k)P(k)} && (0.2)\\
&= \frac{P(E \mid j)p_j}{\sum_{k=1}^{n} P(E \mid k)p_k} && (0.3)
\end{aligned}
$$

Numerators. We'll need the probabilities $P(E \mid j)$ is 1 if $j \neq i$ because we can't find the coin in i if it is in box j.
$P(E \mid j) = (1 - a_i)$ if $j = i$, i.e. 1 minus the probability of finding it in box i when we search box i if it is in box i.
Denominator

$$\sum_{k=1}^{n} P(E \mid k) p_k = (1 - a_i) p_i + \sum_{k \neq i} p_k \tag{0.4}$$

$$= (1 - a_i) p_i + (1 - p_i) \tag{0.5}$$

$$= (1 - a_i p_i) \tag{0.6}$$

Or the only way E does not happen is that the coin is in i and you find it when you search. The probabilities can then be plugged into Bayes' formula to get the two expressions.