Ideas of mathematical proof. Practical class week 25

Note that solutions must be by the methods specified in the problems – even if you can see another solution, even if an easier one.

7.1. Prove by contradiction that $\sqrt[3]{3} \notin \mathbb{Q}$.

7.2. Let $P(x, y)$ be the predicate $x > y$, and $Q(x, y)$ the predicate $x^2 < y^2$, with universe of discourse $\mathscr{U} = \mathbb{R}$.

(a) What is the truth value of $P(x, y) \Rightarrow Q(x, y)$

(1) for $x = -3, y = 2$? (2) for $x = 2, y = -1$?

(b) What is the truth value

(1) of $\forall x \forall y (P(x, y) \Rightarrow Q(x, y))$? (2) of $\forall y \exists x (P(x, y) \Rightarrow Q(x, y))$?

7.3. Let x denote a student in UoL, and y a song of Beatles. Let $P(x, y)$ denote the predicate "x likes y".

(a) Express in symbolic form using quantifiers the statement "Some students in UoL do not like any of Beatles' songs".

(b) Apply the rules for negations of quantified statements to express the negation of the expression obtained in part (a) so that negation sign is not before quantifiers.

(c) Translate into natural language the formula obtained in part (b).

7.4. Let $\mathscr{U} = \mathbb{R}$. Determine which of these statements are true:

(a) $\forall x \, \forall y \, (x^2 < 2xy)$;

(b) $\exists x \, \forall y \, (x^2 < 2xy)$;

(c) $\forall x \, \exists y \, (x^2 \leqslant 2xy)$;

(d) $\forall y \, \exists x \, (x^2 < 2xy)$.

7.5. Recall the theorem that $\aleph_0$ is the smallest infinite cardinal; in other words: if there is an injective mapping $B \to A$ and $|A| = \aleph_0$, then B is countable. Use this theorem to show that any set consisting of pairwise disjoint discs in $\mathbb{R}^2$ (each of radius > 0) is countable.

[*Hint*: see the solution of question 5.5 in Practical W22 and use the fact that $|\mathbb{Q} \times \mathbb{Q}| = \aleph_0$.]