Question 1

(a) Use mathematical induction to prove that $5^k > 15k$ for any positive integer $k \geqslant 3$. [8 marks]

(b) Solve the system of simultaneous inequalities

$$\begin{aligned} x^2 - 2x &\geqslant 0 \\ x^2 - x - 6 &< 0 \end{aligned}$$

and represent the solution as a union of intervals. [8 marks]

(c) Let $\sim$ be a relation on the set $A = \{1, 2, 3, 4, 5, 6, 7\}$ defined as

$a \sim b$ if and only if $a + b$ is even.

(i) Prove that $\sim$ is an equivalence relation;

(ii) depict this relation on the diagram as a subset of $A \times A$;

(iii) list the elements of the corresponding equivalence class $[3]$ of the element 3. [9 marks]

Question 2

(a) Use mathematical induction to prove that $2^{2n+1}+1$ is divisible by 3 for any positive integer n. [8 marks]

(b) Determine the truth tables for the following statements and indicate which of them (if any) are tautologies or contradictions:

(i) $(P \vee Q) \Rightarrow (P \wedge \neg Q)$;

(ii) $(\neg Q \Rightarrow P) \vee (\neg P \vee Q)$. [8 marks]

(c) (i) State the definition of a limit of a function $\lim_{x\to+\infty} f(x) = L$ as $x \to +\infty$.

(ii) Prove from first principles, by verifying the definition that $\lim_{x\to+\infty} \frac{1}{\ln x} = 0$. [9 marks]

Question 3

(a) Let a_n be a sequence defined recursively as $a_1 = 1,\ a_2 = 5$, and $a_i = 3a_{i-1} + 4a_{i-2} + 2$ for $i \geqslant 3$.
Use mathematical induction to prove that $a_n = \dfrac{4^n - 1}{3}$ for all positive integers n.
[*Hint*: first check the cases $n = 1$ and $n = 2$.] [8 marks]

(b) For each of the following mappings, determine whether it is (1) injective, (2) surjective, giving reasons for your answers:

(i) $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$, where $f((a, b)) = (a + b,\ a - b)$;

(ii) $g : \mathbb{R} \to (0, 1]$, where $g(x) = \dfrac{1}{x^2 + 1}$;

(iii) $h : \mathscr{P}(\mathbb{N}) \to \mathscr{P}(\mathbb{Z})$, where $h(X) = \{-a \mid a \in X\}$.
(Recall that $\mathscr{P}(A)$ denotes the set of all subsets of a set A.) [8 marks]

(c) Prove by contradiction that $\log_{10} 7$ is an irrational number. [9 marks]

Question 4

(a) Use the properties of operations on sets to show that

$\overline{A \cap (\overline{A} \cup B)} = \overline{A} \cup \overline{B}$. [8 marks]

(b) Let $\mathscr{U} = \mathbb{R}$. Determine which of these statements are true giving reasons to your answers:

(1) $\forall x \, \forall y \; ((x < y) \Rightarrow (x^2 < y^2))$;

(2) $\exists x \, \forall y \; ((x < y) \Rightarrow (x^2 < y^2))$;

(3) $\exists x \, \exists y \; ((x > y) \wedge (x^2 < y^2))$. [8 marks]

(c) State and prove the theorem about the limit of the sum of two sequences (a_n) and (b_n) each of which has a finite limit as $n \to \infty$. [9 marks]