UNIVERSITY OF
LINCOLN
UNITED KINGDOM

Time Constrained Assessment Examination

School	School of Mathematics and Physics
Module Title	Ideas of Mathematical Proof
Module Code	MTH1003M
Module Coordinator	E. Khukhro
Duration of Assessment	3 hours
Date	May/June 2022

General Instructions to Candidates

1. In sitting this examination you agree to **comply** with the University of Lincoln Code of Conduct in Examinations.
2. You **must** submit your answers as a PDF to Turnitin on Blackboard **before** the submission time: failure to do so will be classified as misconduct in examinations.
   We strongly recommend to having submitted your work at least 15 minutes prior to the deadline.
3. You **must** also send a copy of your work as an attachment to the email address **SMPsubmissions@lincoln.ac.uk** at the same time. You must place the Module Code and your Student Id in the Subject Field of the Mail.
4. This assessment is an **open resource format**: you may use online resources, lecture and seminar notes, text books and journals.
5. **No collaboration or interaction** with other candidates or individuals using any means of communication or device is permitted during online examinations.
6. All work will be **subject to plagiarism and academic integrity checks**. In submitting your assessment you are claiming that it is your own original work; if standard checks suggest otherwise, Academic Misconduct Regulations will be applied.
7. You must **show all your working**, and submit only one version of your solution. You may cross out any failed attempts and they will be ignored.
8. **The duration of the Time Constrained Assessment will vary for those students with Personal Academic Study Support (PASS).** Extensions do not apply, but Extenuating Circumstances can be applied for in the normal way.

Module Specific Instructions to Candidates:

QUESTIONS TO ANSWER: **Answer ALL FOUR questions.**

MARKING SCHEME: **Each question carries TWENTY FIVE (25) marks**

Question 1

(a) Solve the system of simultaneous inequalities [8 marks]

$$\begin{aligned} x^2 - 6x + 8 &\geqslant 0 \\ x^2 - 4x + 3 &> 0 \end{aligned}$$

and represent the solution as a union of intervals.

(b) Let $\sim$ be a relation on the (x, y) coordinate plane $\mathbb{R} \times \mathbb{R}$ defined as [9 marks]

$$(x_1, y_1) \sim (x_2, y_2) \text{ if } x_1^2 + y_1^2 = x_2^2 + y_2^2.$$

Prove that $\sim$ is an equivalence relation and indicate the equivalence classes of the elements $(0, 0)$ and $(-3, 4)$ by pictures on the coordinate plane.

(c) State and prove the theorem about the limit of the sum of two functions each of which has a finite limit as $x \to a$. [8 marks]

Question 2

(a) Determine the truth tables for the following statements and indicate which of them (if any) are tautologies or contradictions:

(i) $(P \wedge Q) \Rightarrow (\neg P \vee Q)$; [4 marks]

(ii) $(P \Rightarrow \neg Q) \vee (P \wedge Q)$. [4 marks]

(b) (i) State the definition of a finite limit of a sequence $\lim_{n\to\infty} a_n = a$. [4 marks]

(ii) Prove from first principles, by verifying the definition, that [5 marks]

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}+1} = 0.$$

(c) Let S be the set of all infinite sequences of the form $(a_1, a_2, \ldots)$, where each of the a_i is either letter X or letter Y. Use Cantor's diagonalization method to prove by contradiction that S is uncountable. [8 marks]

Question 3

(a) Let a_n be a sequence defined recursively as $a_1 = 4,\ \ a_2 = 6$, and $a_i = a_{i-1} + 2a_{i-2} - 4$ for $i \geqslant 3$. [8 marks]
Use mathematical induction to prove that $a_n = 2^n + 2$
for all positive integers n.
[*Hint*: first check the cases $n = 1$ and $n = 2$.]

(b) Given that A and B are sets, use the properties of operations on sets to simplify the expression [8 marks]

$$\overline{\overline{A} \cup (\overline{B} \cap A)}.$$

(c) For each of the following mappings, determine whether it is
(1) injective, (2) surjective, giving reasons to your answers.

(i) $f : \mathbb{N} \times \mathbb{N} \to \mathbb{Q},\ \ f((a,b)) = \dfrac{a}{b}$ [3 marks]

(ii) $g : (0,\infty) \to (0,\infty),\ \ g(x) = \dfrac{1}{x^2}$; [3 marks]

(iii) $h : A \to A$, where $A = \mathscr{P}(\{u,v,w\})$ is the set of all subsets of $\{u,v,w\}$ and $h(X) = X \cup \{v\}$ for $X \in A$. [3 marks]

Question 4

(a) Use mathematical induction to prove that $4^k > 10k$ for any positive integer $k \geqslant 3$. [8 marks]

(b) Prove by contradiction that $\sqrt{3} \notin \mathbb{Q}$. [8 marks]

[*Hint*: Assume as known the following fact:
if a^2 is divisible by 3 for an integer a, then a is also divisible by 3.]

(c) Demonstrate that the set of positive integers $\mathbb{N}$ and the Cartesian product $\{1, 2\} \times \mathbb{N}$ have the same cardinality by exhibiting a bijective mapping [9 marks]

$f : \mathbb{N} \to \{1, 2\} \times \mathbb{N}$.