UNIVERSITY OF
LINCOLN
UNITED KINGDOM

Time Constrained Assessment Examination

School	**School of Mathematics and Physics**
Module Title	**Ideas of Mathematical Proof**
Module Code	**MTH1003M**
Module Coordinator	**E. I. Khukhro**
Duration of Assessment	**3 hrs**
Date	**2^{nd} of June 2021**
Release Time	**15:00**
Submission Time	**18:00**

General Instructions to Candidates

1. In sitting this examination you agree to **comply** with the University of Lincoln Code of Conduct in Examinations.
2. You **must** submit your answers as a PDF to Turnitin on Blackboard **before** the submission time: failure to do so will be classified as misconduct in examinations. **We strongly recommend that you have your work submitted at least 15 minutes prior to the deadline.**
3. You **must** also send a copy of your work as an attachment to the email address **SMPsubmissions@lincoln.ac.uk** at the same time. You must place the Module Code and your Student Id in the Subject Field of the Mail.
4. This assessment is an **open resource format**: you may use online resources, lecture and seminar notes, text books and journals.
5. **No collaboration or interaction** with other candidates or individuals using any means of communication or device is permitted during online examinations.
6. All work will be **subject to plagiarism and academic integrity checks**. In submitting your assessment you are claiming that it is your own original work; if standard checks suggest otherwise, Academic Misconduct Regulations will be applied.
7. You must **show all your working**, and submit only one version of your solution. You may cross out any failed attempts and they will be ignored.
8. **The duration of the Time Constrained Assessment will vary for those students with Personal Academic Study Support (PASS)**. Extensions do not apply, but Extenuating Circumstances can be applied for in the normal way.

Module Specific Instructions to Candidates

QUESTIONS TO ANSWER: **Answer ALL FOUR questions.**

MARKING SCHEME: **Each question carries TWENTY FIVE (25) marks**

Question 1

(a) Solve the system of simultaneous inequalities [8 marks]

$$\begin{aligned} x^2 + x - 6 &> 0 \\ x^2 - 25 &< 0 \end{aligned}$$

and represent the solution as a union of intervals.

(b) Let $\sim$ be a relation on the (x, y) coordinate plane $\mathbb{R} \times \mathbb{R}$ defined as [9 marks]

$$(x_1, y_1) \sim (x_2, y_2) \text{ if } |x_1| + y_1 = |x_2| + y_2.$$

Prove that $\sim$ is an equivalence relation and indicate the equivalence classes of the elements $(0, 0)$ and $(1, -2)$ by pictures on the coordinate plane.

(c) State and prove the theorem about the limit of the sum $(a_n + b_n)$ of two convergent sequences (a_n) and (b_n). [8 marks]

Question 2

(a) Determine the truth tables for the following statements and indicate which of them (if any) are tautologies or contradictions:

(i) $(P \Rightarrow Q) \vee (P \wedge (\neg Q))$; [4 marks]

(ii) $(P \vee Q) \Rightarrow (P \wedge (\neg Q))$. [4 marks]

(b) (i) State the definition of a finite limit of a sequence $\lim_{n\to\infty} a_n = a$. [4 marks]

(ii) Prove from first principles, by verifying the definition, that [5 marks]

$$\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}.$$

(c) Let S be the set of all infinite sequences of the form $(a_1, a_2, \ldots)$, where each of the a_i is either 1 or 2. Use Cantor's diagonalization method to prove by contradiction that S is uncountable. [8 marks]

Question 3

(a) Let a_n be a sequence defined recursively as $a_1 = 5,\ a_2 = 7$, and $a_i = a_{i-1} + 2a_{i-2} - 6$ for $i \geqslant 3$. [8 marks]

Use mathematical induction to prove that $a_n = 2^n + 3$ for all positive integers n.

[*Hint*: first check the cases $n = 1$ and $n = 2$.]

(b) Given that A and B are sets, use the properties of operations on sets to simplify the expression [8 marks]

$$\overline{\overline{A} \cap (B \cup A)}.$$

(c) For each of the following mappings, determine whether it is (1) injective, (2) surjective, giving reasons to your answers.

(i) $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N},\ f((a, b)) = 2^a \cdot 3^b$; [3 marks]

(ii) $g : [0, \infty) \to (0, 1],\ g(x) = \dfrac{1}{x^2 + 1}$; [3 marks]

(iii) $h : A \to A$, where $A = \mathscr{P}(\{a, b, c\})$ is the set of all subsets of $\{a, b, c\}$ and $h(X) = X \cap \{b, c\}$. [3 marks]

Question 4

(a) Use mathematical induction to prove that $3^n > 20n$ for any positive integer $n \geqslant 4$. [8 marks]

(b) Let R be the relation on the set $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ defined as aRb if b is divisible by a.

(i) Prove that R is an order relation. [5 marks]

(ii) Depict the relation R as a subset on the diagram of the Cartesian product $A \times A$. [4 marks]

(c) Demonstrate that the sets of positive integers $\mathbb{N}$ and all integers $\mathbb{Z}$ have the same cardinality by exhibiting a bijective mapping $f : \mathbb{Z} \to \mathbb{N}$. [8 marks]