Algebra – Practical for week 1

It makes sense that you work on the questions in the given order, but questions marked with * are likely to be harder than the rest, so you may leave those to the end.
When you get stuck on a question, or you are not sure that you are proceeding correctly, there is no shame in reading the solution provided, or to ask me for help. In fact, occasionally some solution (especially of the starred questions) is a way for me to teach you something but I want you to have a go at it first.
In any case, please always read my solutions afterwards, even of questions that you know how to answer, because often solutions contain comments, further information such as alternate ways of proceeding, etc.
There are usually more questions than you might be able to answer in a 1-hour practical, so take the rest as homework, or start to work on them ahead of the practical.

1.1.

(a) Factorise 300 (completely, hence as a product of prime numbers).
(b) Find all positive divisors of 300, and arrange them in a Hasse diagram.
(c) How many positive divisors does 300 have?

1.2.

(a) Factorise $n = 75600$ into a product of prime factors.
(b) Find the number of distinct *positive* divisors of n. (There is no need to compute them all in order to answer this question.)
(c) Find the number of distinct *odd* positive divisors of n.
(d) Find the three *largest* positive divisors of n (meaning largest in numerical value here, so with respect to $\leq$ rather than $|$).
Note that the Hasse diagram would be quite big in this case, but it may still help you (or possibly not...) to draw just the top part of it.

1.3. Use the Euclidean algorithm to find the greatest common divisor of 300 and 221.

1.4. Use the Euclidean algorithm to compute the greatest common divisor of 299 and 221. Now use the result to find the (complete) factorisations of 299 and 221.

1.5*. Apply the Euclidean algorithm to find the greatest common divisor of $200000011 = 2 \cdot 10^8 + 11$ and $100000003 = 10^8 + 3$.

1.6*. If n is any positive integer, show that $10^n + 1$ and $2 \cdot 10^n + 1$ are coprime.

The following questions will test your understanding of the definition of divisibility.

1.7. Show that if the integer a divides both integers b and c, then a divides each of $b+c$, $b-c$, and bc.

1.8. Use the definition to show that *to divide* is a *transitive* relation, which means the following: if $a \mid b$ and $b \mid c$, then $a \mid c$.

1.9*. Consider the following statement:

$$\textit{If } a, b \in \mathbb{Z}, \textit{ and } b \mid a, \textit{ then } b \leq a.$$

(a) Show that the statement is *false* (in the exact way it is written here).
Hint: In order to prove that a statement is false, it is enough to find a *counterexample,* that is, one particular instance where it fails; in this case, two particular numbers a and b for which it fails.
(b) Make a slight change to this false statement so it becomes a *true* statement.