Even and odd functions

- To see if f is odd, even or neither, replace every instance of x with $-x$.
- If you get back the original function, f is even. If you find the negative of the original function, f is odd.
- If neither of these things is true, f is neither odd nor even.

Example $\quad g(x) = 2x^3 + x$

$\Rightarrow$ "implies that"

$$g(-x) = 2(-x)^3 + (-x)$$
$$= -2x^3 - x = -g(x)$$

Exponential functions

- An exponential function is a function of the form $y = a^x$, where a is a real constant
- The domain is the whole real line, $\mathbb{R}$, and the range is $(0, \infty)$: the graph never touches the x-axis.
- The standard exponential function is $y = e^x = \exp(x)$, where $e \approx 2.718\cdots$ (an irrational number)
- The function $\exp(x)$ is equal to its own derivative, and is defined by the power series

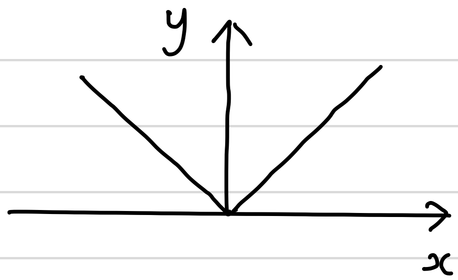
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

- Its inverse is the natural logarithm, $y = \ln x$, whose domain and range are $(0, \infty)$ and $\mathbb{R}$ respectively.

Piecewise-defined functions

① The absolute value function is given by

$$f(x) = |x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$



Domain : $(-\infty, \infty) = \mathbb{R}$
Range : $[0, \infty)$

② The signum function

$$f(x) = \operatorname{sgn}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

The range of the signum function is $\{-1, 0, 1\}$: the set of the three numbers -1, 0 and 1. Its domain is $\mathbb{R}$.

Hyperbolic functions

- These are even and odd combinations of e^x and e^{-x}.

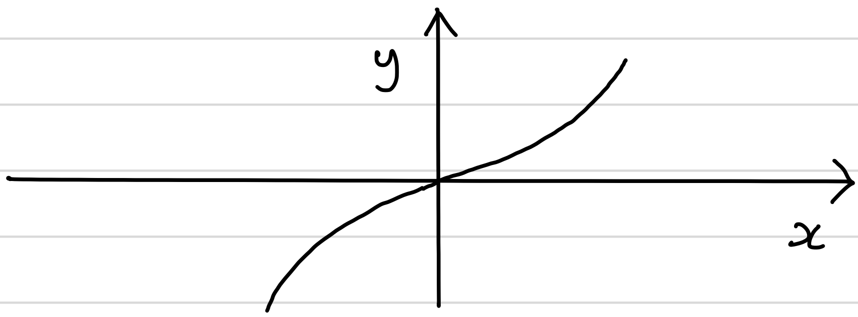
$\sinh x = \dfrac{e^{x} - e^{-x}}{2}$: hyperbolic sine

$\cosh x = \dfrac{e^{x} + e^{-x}}{2}$: hyperbolic cosine

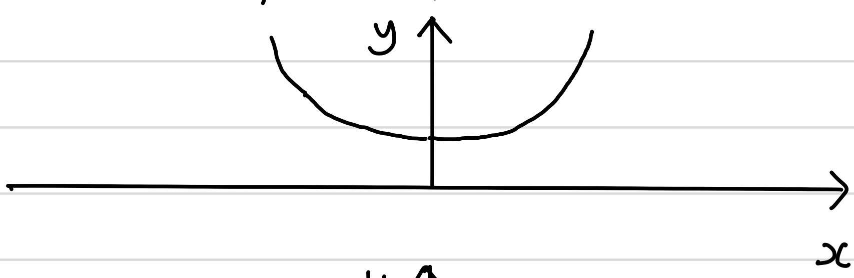
$\tanh x = \dfrac{\sinh x}{\cosh x}$: hyperbolic tangent

$\operatorname{csch} x = \dfrac{1}{\sinh x}$ $\qquad$ $\operatorname{sech} x = \dfrac{1}{\cosh x}$

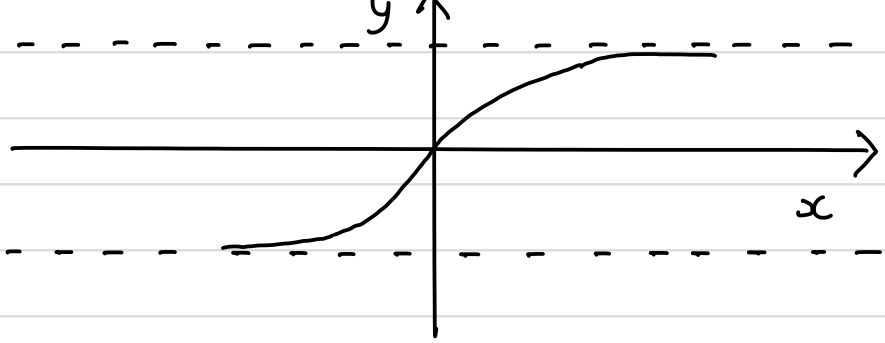
$\coth x = \dfrac{\cosh x}{\sinh x}$



$y = \sinh x$



$y = \cosh x$



$y = \tanh x$

The domains of $\sinh x$, $\cosh x$ and $\tanh x$ are all equal to $\mathbb{R}$: all the real numbers. There are no intervals where they are not defined, as none of their definitions involve prohibited operations such as division by zero.

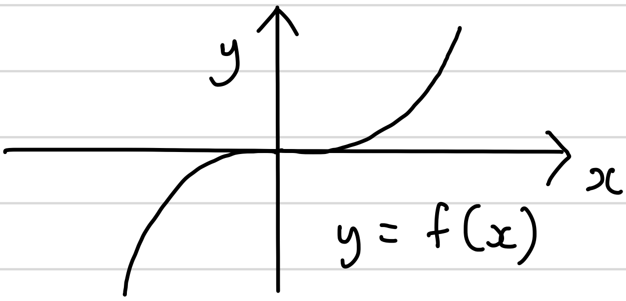
Composition of two functions

Let both f and g be functions of real variables. Their composition is a rule that assigns a value z in the range of g to each value x in the domain of f: $y = f(x)$, $z = g(y)$

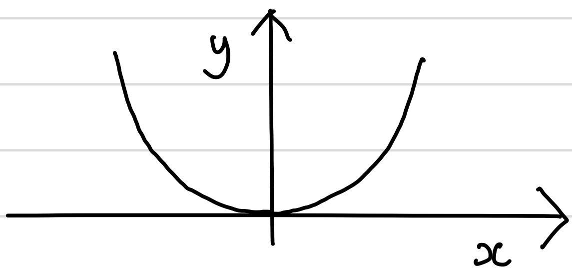
$$g \circ f(x) = g(f(x))$$

Example $f(x) = x^3$, $g(y) = |y|$

Then, $h(x) = g \circ f'(x) = |f(x)| = |x^3|$



Domain of f: $\mathbb{R}$
Range of f: $\mathbb{R}$



$$y = h(x) = g \circ f(x) = |x^3|$$

Domain of h: $\mathbb{R}$ Range of h: $[0, \infty)$

Complex numbers

The solution of a quadratic equation $ax^2 + bx + c$ can be found using

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If the term $b^2 - 4ac < 0$, we usually say that the quadratic has no (real) roots.

<u>Example</u> The formula above gives the roots of the quadratic equation

$$z^2 - 4z + 5 = 0 \quad \text{as} \quad \frac{2 \pm \sqrt{-4}}{2}.$$

However, if we define $i = \sqrt{-1}$, we can write these roots as

$$\frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2\sqrt{-1}}{2} = 2 \pm \sqrt{-1} = 2 \pm i$$

This is an example of a complex number. It has a real part and an imaginary part.

More generally, we can write

$$z = a + bi, \quad \text{where } a, b \in \mathbb{R}$$

a and b are the real and imaginary parts, respectively, of z, often denoted by $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$.

- If $a = 0$, then z is purely imaginary.
  If $b = 0$, then z is real.
- A complex number written as the sum of a real and an imaginary term is in <u>standard form</u>.
- We denote the set of all complex numbers by $\mathbb{C}$, so that $z \in \mathbb{C}$.
- Every quadratic equation has roots in $\mathbb{C}$.

Addition and subtraction

- The real and imaginary parts are treated separately.
- Example: If $z_1 = 2 + 3i$ and $z_2 = 1 - 2i$, then

$$z_1 + z_2 = (2+1) + (3-2)i = 3 + i$$

and
$$z_1 - z_2 = (2-1) + (3-(-2))i = 1 + 5i$$

Multiplication

- The essential fact to remember is that $i^2 = -1$ (since $i = \sqrt{-1}$).

Example

With z_1 and z_2 as above, we have

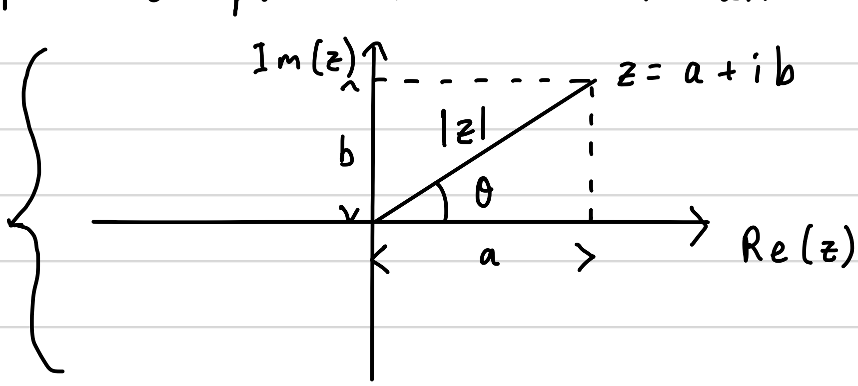
$$\begin{aligned} z_1 z_2 &= (2+3i)(1-2i) \\ &= 2 - 4i + 3i - 6(i^2) \end{aligned}$$

$$= 2 - i - 6(-1)$$
$$= 2 - i + 6$$
$$= 8 - i$$

The Argand diagram

We often plot complex numbers on an Argand diagram:

the complex plane



Definitions and properties

- The complex conjugate of $z = a + ib$ is $\bar{z} = a - ib$ (sometimes written $z^* = a - ib$)
- The modulus of z is $|z| = r = (a^2 + b^2)^{1/2}$
- We also have that $|z|^2 = z\bar{z}$, since

$$z\bar{z} = (a + ib)(a - ib)$$
$$= a^2 - iba + iba - (i)^2 b^2$$
$$= a^2 + b^2$$

- The argument of $z = a + ib$ is $\arg(z) \equiv \theta$.

Note: the argument is not unique. An integer multiple of 2π can be added to θ without changing z, as this just takes us back to the same point on the Argand diagram. Usually, we take $0 \leq \theta < 2\pi$.

We also have that $\theta = \tan^{-1}\left(\frac{Im(z)}{Re(z)}\right) = \tan^{-1}\left(\frac{b}{a}\right)$

<u>Example</u> With $z = \sqrt{3} + i$, we have that $\bar{z} = \sqrt{3} - i$

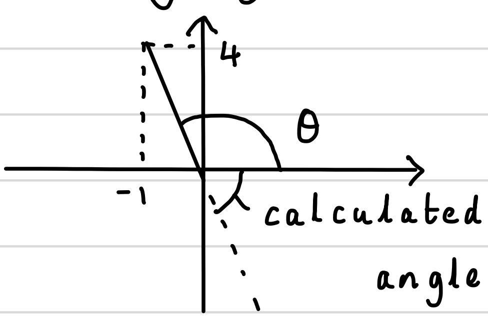
$|z| = r = [(\sqrt{3})^2 + 1^2]^{1/2} = 2$

$\arg(z) = \tan^{-1}(1/\sqrt{3}) = \pi/6$

<u>Note</u>: it is important to remember that calculating θ from $\tan^{-1}(b/a)$ can give an answer in the wrong quadrant. It can be useful to sketch an Argand diagram to make sure that you have calculated the correct angle.

<u>Example</u> With $z = -1 + 4i$, we have that $\tan^{-1}[4/(-1)] \approx -1.3258$ rad.

However, drawing an Argand diagram shows that this lies in the wrong quadrant:



We can move into the correct quadrant by adding π to our result, giving $\theta \approx 1.8158$ rad.

<u>Division</u>

The complex conjugate introduced above is important in division. For example, to calculate the quotient

$$\frac{7-4i}{4+3i}$$

we multiply the numerator + denominator by the complex conjugate of the denominator, $4-3i$.

This gives $\dfrac{(7-4i)(4-3i)}{(4+3i)(4-3i)} = \dfrac{28-37i-12}{16+9} = \dfrac{16}{25} - \dfrac{37}{25}i$

The fact that $z\bar{z} = a^2 + b^2$ has allowed us to make the denominator entirely real and write the quotient in standard form.

<u>Polar form of complex numbers</u>

From the Argand diagram, we can see that

$$\begin{aligned} z &= r\cos\theta + i\, r\sin\theta \\ &= r(\cos\theta + i\sin\theta) \end{aligned}$$

This is the polar form of a complex number.

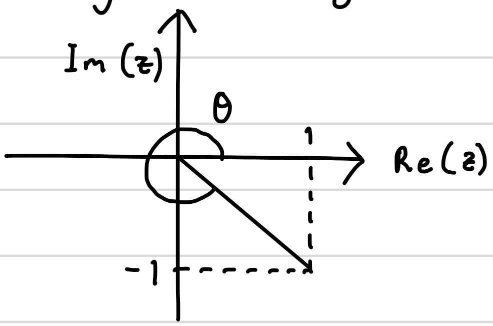
<u>Example</u> Write $z = 1-i$ in polar form.

(1) Calculate the modulus, $|z| = r = [1^2 + 1^2]^{1/2} = \sqrt{2}$

(2) Calculate $\theta = \tan^{-1}(b/a) = \tan^{-1}(-1/1)$

$$= \tan^{-1}(-1) = \frac{3\pi}{4} \text{ or } \frac{7\pi}{4}$$

(3) Draw an Argand diagram...



... to see that $\theta = \frac{7\pi}{4}$

The polar form of z is then

$$z = \sqrt{2}\,[\cos(7\pi/4) + i\sin(7\pi/4)]$$

Exponential form of a complex number

Recall: the exponential, sine and cosine functions can be written as power series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

If we define $e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots$

then $e^{i\theta} = 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \cdots$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \cdots\right)$$

$$= \cos\theta + i\sin\theta$$

<u>Euler's identity</u>.

This gives us another representation of a complex number: the exponential form

$$\boxed{z = re^{i\theta}, \quad \text{with } e^{i\theta} = \cos\theta + i\sin\theta}$$

Note: $|e^{i\theta}| = [(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)]^{1/2}$

$$= [\cos^2\theta + \sin^2\theta]^{1/2} = 1$$

and $e^{i\theta}$ represents all points on a circle of unit radius in the Argand diagram.