MTH 1002 — Calculus

Assessments

- Exam : 60% of module mark
- WebAssign assignments: 8% of module mark
- Coursework : 7% of module mark
- Test : 25% of module mark

Reading
- Calculus, by J. Stewart
- Calculus, by M. Spivak
- any maths for physics / engineering textbook e.g. Engineering Mathematics by K. A. Stroud

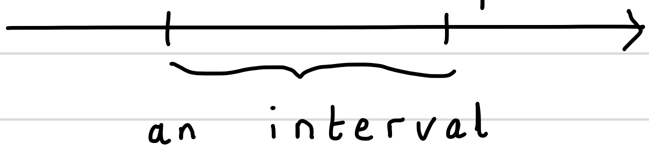
Course content

- Functions
- Complex numbers
- Limits and continuity
- Differentiation
- Curve sketching
- Taylor series
- Integration and the fundamental theorem of calculus
- Parametric equations and polar coordinates

- Multivariable calculus

Intervals

An interval is a connected portion of the real line.

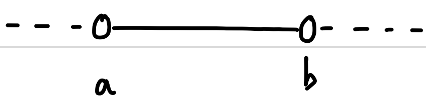


Finite intervals

Definition: A subset I of the real line is called an interval if it contains at least two numbers and every number lying between them; that is, if $x, y \in I$ and $z \in \mathbb{R}$, $x < z < y$, then $z \in I$.

- If we suppose that $a, b \in \mathbb{R}$, we can consider the following kinds of interval:

- Open interval: contains neither endpoint



$$(a, b) = \{x \in \mathbb{R} : a < x < b\}$$

"such that" strict inequalities

- Closed interval: contains both endpoints

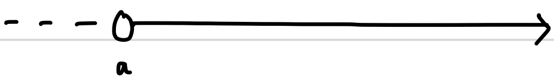
$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$$

- <u>Half-open interval</u> : contains one endpoint

$(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$

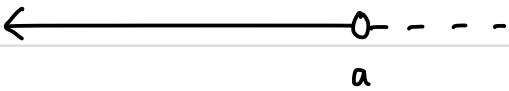
<u>Semi-infinite intervals</u>

For a real value $a \in \mathbb{R}$, we have the following semi-infinite intervals:

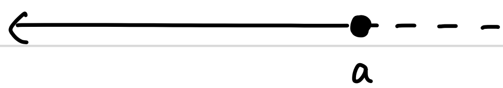


$(a, \infty) = \{x \in \mathbb{R} : x > a\}$

$[a, \infty) = \{x \in \mathbb{R} : x \geq a\}$



$(-\infty, a) = \{x \in \mathbb{R} : x < a\}$



$(-\infty, a] = \{x \in \mathbb{R} : x \leq a\}$

<u>Functions of a real variable</u>

- A function is a rule for transforming an object into another object.
e.g. the area of a circle is a function of its radius, $A = \pi r^2$.

<u>Definition</u> : A function f from a set X to a set Y is a rule that assigns a unique element $y \in Y$ to each element $x \in X$.

X is the domain of the function. It is the set of all values to which the function can be applied. Sometimes, the domain is specified.

- If you are asked to find the domain of a function, you will have to exclude values of x to which it cannot be applied.

Example The domain of the function $y = \sqrt{x}$ is $[0, \infty)$, because we cannot take the square root of a negative number.

- The range of the function is the set of all possible values of f as x varies throughout the domain.

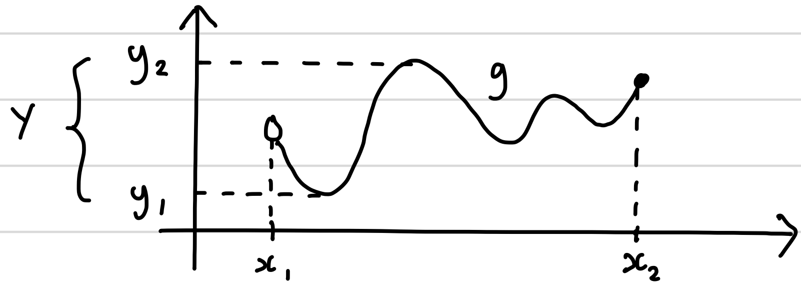
Example: the range of the function $y = \sqrt{x}$ is $[0, \infty)$, because we take the positive root unless otherwise specified.

Example What are the domain and range of the function $y = \sqrt{10 - x}$?

- Domain: $(-\infty, 10]$ (provided $x \leqslant 10$, we can take the square root)
- Range: $[0, \infty)$ (as for $y = \sqrt{x}$)

Definition: the graph of a function f is the set of all Cartesian coordinates where x is in the domain X and $y = f(x)$.

Graph of a continuous function



Graph: g

Domain:

$$X = \{x_1 < x \leq x_2\} = (x_1, x_2]$$

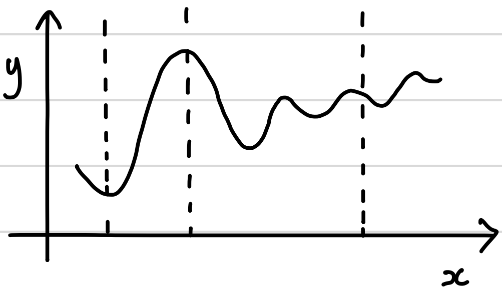
Range:

$$Y = \{y_1 \leq y \leq y_2\} = [y_1, y_2]$$

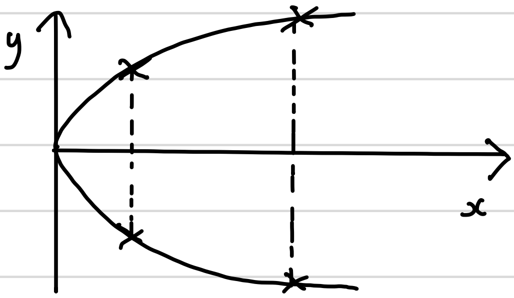
The vertical line test

Recall: a function f from a set X to a set Y is a rule that assigns a unique element $y \in Y$ to each element $x \in X$.

We can represent this graphically by the vertical line test.

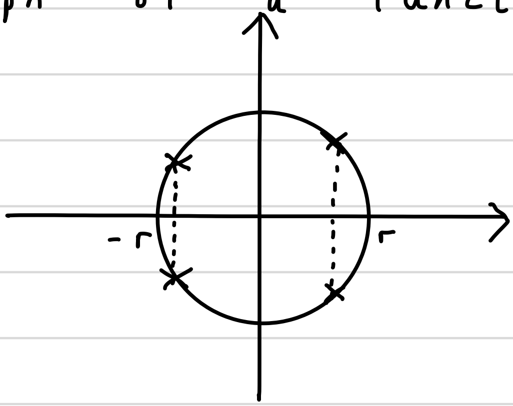


- This is the graph of a function: each line intersects the graph only once.



- This is not the graph of a function: each line intersects the graph twice.

Example A circle in the (x, y) plane is not the graph of a function:



We need to define two functions:

Equation for whole circle : $x^2 + y^2 = r^2$
Top semicircle : $y = \sqrt{r^2 - x^2}$
Bottom semicircle : $y = -\sqrt{r^2 - x^2}$

Polynomial functions

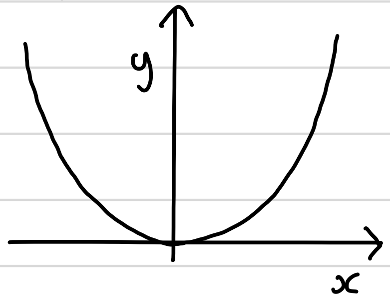
- A polynomial in x of degree n has the form

$$f(x) = a_n x^n + \dots + a_2 x^2 + a_1 x + a_0 = \sum_{j=0}^{n} a_j x^j$$

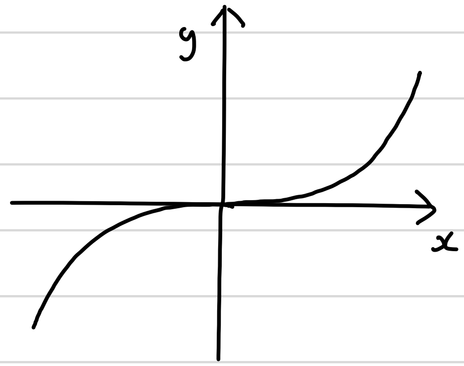
where the quantities a_j are constant coefficients and $a_n \neq 0$.
- The domain of a polynomial is all the real numbers, because
  - there are only integer powers of x, so there is no risk of taking the root of a negative number.
  - all powers of x are non-negative, so there is no risk of division by zero.

Graphs of the power x^n

- Graphs of $y = x^n$ for even $n \geq 2$ have the general form



- For odd $n \geq 3$, the graph of $y = x^n$ looks like:

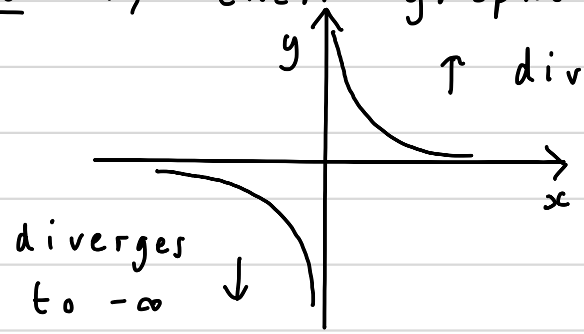


Rational functions

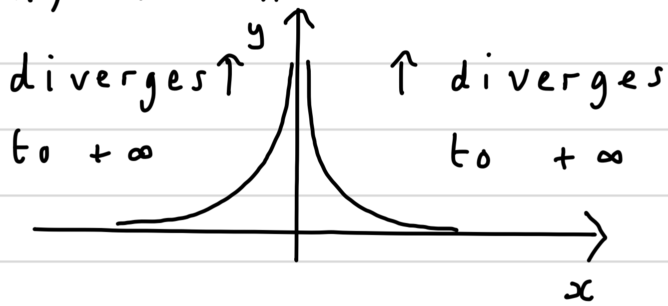
- These have the form $\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials.

- Simple examples of rational functions are the functions $f(x) = 1/x^n$, where n is a positive integer

- For <u>odd</u> n, their graphs have the form



- For <u>even</u> n, we have



- These are examples of functions whose domain does not include the whole real line.
- The domain of $f(x) = 1/x^n$ for both even and odd n is $\{x \in \mathbb{R} : x \neq 0\} = (-\infty, 0) \cup (0, \infty)$ ($x = 0$ must be excluded, as f is not defined here.)
- The range of $f(x) = 1/x^n$ is $\{y \in \mathbb{R} : y \neq 0\} = (-\infty, 0) \cup (0, \infty)$ for odd n, and $\{y \in \mathbb{R} : y > 0\} = (0, \infty)$ for even n.

<u>Example</u> (Exam 2020): Write down the domain and range of $f(x) = \dfrac{1}{(x-3)^4}$.

- Division by zero at $x = 3 \Rightarrow x = 3$ is excluded from the domain.
- Domain is $\{x \in \mathbb{R} : x \neq 3\} = (-\infty, 3) \cup (3, \infty)$

- Range: $f(x)$ can produce arbitrarily small numbers (as $x \to \pm\infty$) and arbitrarily large numbers (as $x \to 3$).
- The power is even, so these numbers are positive and the range is $(0, \infty)$.

Trigonometric functions

- Basic functions: sin and cos. These are $\pi/2$ out of phase, but are otherwise identical.
- Both have domain $(-\infty, \infty)$ and range $[-1, 1]$.
- They are defined by the power series

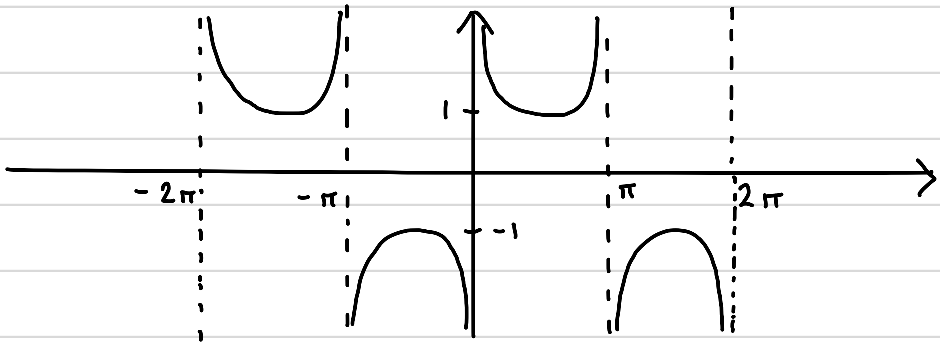
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

- Their quotient, $\tan x = \sin x / \cos x$, is undefined when $\cos x = 0$ i.e. at $x = \pi/2, 3\pi/2, 5\pi/2, \dots,$ or $x = (n + 1/2)\pi$, $n \in \mathbb{Z}$
- Its domain is then $\{x \in \mathbb{R} : x \neq (n + \frac{1}{2})\pi, \ n \in \mathbb{Z}\}$
- Its range is all the real numbers, $y \in \mathbb{R}$.

Example: Sketch the graph of $\operatorname{cosec} x = 1/\sin x$, and state its domain and range.

($\operatorname{cosec} x = \csc x$)



Domain: $\{x \in \mathbb{R} : x \neq n\pi,\ n \in \mathbb{Z}\}$

Range: $\{y \in \mathbb{R} : y \leq -1 \text{ or } y \geq 1\}$

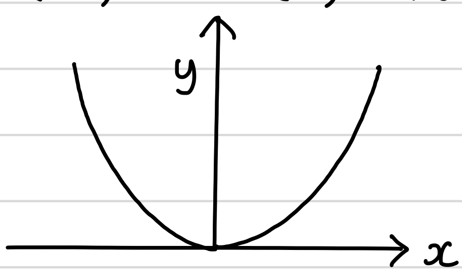
or $(-\infty, -1] \cup [1, \infty)$

- The trigonometric functions are examples of periodic functions: they repeat themselves in successive intervals.

Definition. A function is periodic if there is a $p > 0$ such that $f(x+p) = f(x)$ for all x in the domain of f. The smallest such number is called the period of f.

Even and odd functions

Even function: $f(-x) = f(x)$ for all x in the domain of f.



Odd function: $f(-x) = -f(x)$ for all x in the domain of f.