MTH1001 Algebra – Mock exam

Question 1. Write the integer 1000 (written here in decimal notation) in base 9. [10 marks]

Solution: The digits in base 9, starting from the least significant, are obtained as the remainders of successively dividing first 1000, and then each previous quotient, by 9:

$$\begin{aligned} 1000 &= 9 \cdot 111 + 1 \\ 111 &= 9 \cdot 12 + 3 \\ 12 &= 9 \cdot 1 + 3 \\ 1 &= 9 \cdot 0 + 1 \end{aligned}$$

Hence $(1000)_{10}$ reads $(1331)_9$ in base 9. In other words, $1000 = 1 \cdot 9^3 + 3 \cdot 9^2 + 3 \cdot 9 + 1$.

Question 2. Find all solutions (in the complex numbers) of the following system of equations: [10 marks]

$$\begin{cases} x^3 + y^3 = 13 \\ x + y = 3 \end{cases}$$

Solution: Because $x^3 + y^3 = (x+y)^3 - 3(x+y)xy$, and the value of $x+y$ is known from the second equation, the system is equivalent to

$$\begin{cases} 3^3 - 3 \cdot 3xy = 13 \\ x + y = 3 \end{cases}$$

and, in turn, to

$$\begin{cases} xy = 14/9 \\ x + y = 3 \end{cases}$$

Then x and y are the solutions of the quadratic equation $z^2 - 3z + 14/9 = 0$, or $9z^2 - 27z + 14 = 0$ if we prefer, and so they equal

$$\frac{27 \pm \sqrt{27^2 - 4 \cdot 9 \cdot 14}}{18} = \frac{27 \pm \sqrt{729 - 504}}{18} = \frac{27 \pm 15}{18},$$

which are $7/3$ and $2/3$, in any order. Hence the system has exactly two solutions, which are $(x, y) = (2/3, 7/3)$ and $(x, y) = (7/3, 2/3)$.

Question 3. Use the Euclidean algorithm to find a pair of integers s and t such that [10 marks]

$$331s + 177t = 1.$$

Solution: The Euclidean algorithm reads

$$\begin{aligned} 331 &= 177 \cdot 1 + 154 = 177 \cdot 2 - 23 \\ 177 &= 23 \cdot 7 + 16 = 23 \cdot 8 - 7 \\ 23 &= 7 \cdot 3 + 2 \\ 7 &= 2 \cdot 3 + 1, \end{aligned}$$

where we have taken advantage of using negative remainders if smaller in absolute value. Hence 331 and 177 are coprime, that is, their greatest common divisor equals 1, and so we know from the theory that integers s and t as required exist. Working through the above calculation backwards we find

$$\begin{aligned}1 &= 7 - 2\cdot 3\\ &= 7 - (23 - 7\cdot 3)\cdot 3 = -23\cdot 3 + 7\cdot 10\\ &= -23\cdot 3 + (-177 + 23\cdot 8)\cdot 10 = -177\cdot 10 + 23\cdot 77\\ &= -177\cdot 10 + (-331 + 177\cdot 2)\cdot 77 = -331\cdot 77 + 177\cdot 144.\end{aligned}$$

Hence $s = -77$ and $t = 144$ are integers as required. Another possible pair (of infinitely many) is $s = -77 + 177 = 100$ and $t = 144 - 331 = -187$.

Question 4. Find the smallest positive integer x, if it exists, which satisfies [10 marks]

$$\begin{cases} x \equiv 11 \pmod{14} \\ x \equiv 7 \pmod{20} \end{cases}$$

Solution: The system of congruences has solutions because $(20, 14) = 2$ divides $7 - 11 = -4$. By the Euclidean algorithm we find $2 = -20\cdot 2 + 14\cdot 3$. Multiplying by 2 we obtain $11 - 7 = 4 = -20\cdot 4 + 14\cdot 6$, which can be rearranged as $11 - 14\cdot 6 = 7 - 20\cdot 4$. Hence the common value -73 is one solution of both congruences. Any other solution differs by this by a common multiple of 14 and 20, that is to say, a common multiple of 140. Hence the smallest positive solution is $x = -73 + 140 = 67$.

Question 5. Using the method described in the lectures, find all complex roots of the self-reciprocal polynomial [10 marks]

$$f(x) = x^4 - 2x^3 - 13x^2 - 2x + 1.$$

Solution: After noting that zero is not a root we can divide both sides of the equation $x^4 - 2x^3 - 13x^2 - 2x + 1 = 0$ by x^2, and obtain the equivalent equation

$$x^2 - 2x - 13 - \frac{2}{x} + \frac{1}{x^2} = 0.$$

We can rewrite this as

$$\left(x + \frac{1}{x}\right)^2 - 2\left(x + \frac{1}{x}\right) - 15 = 0,$$

which after setting $y = x + \frac{1}{x}$ reads $y^2 - 2y - 15 = 0$, that is, $(y+3)(y-5) = 0$. Hence the root of $f(x)$ are the solutions of the equation $x + \frac{1}{x} + 3 = 0$, and those of the equation $x + \frac{1}{x} - 5 = 0$. The former are $(-3 \pm \sqrt{5})/2$, and the latter are $(5 \pm \sqrt{21})/2$.

Question 6. to find the rational roots of the polynomial [10 marks]

$$f(x) = 6x^3 + 5x^2 - 10x - 6,$$

and then factorise it over $\mathbb{Z}$. Show all your work.

Solution: According to the Rational Root test, if $r/s \in \mathbb{Q}$ is a root of $f(x)$, with $\gcd(r, s) = 1$, then r divides 6 and s divides 6, hence $r, s \in \{\pm 1, \pm 2, \pm 3, \pm 6\}$. Consequently, the possibilities for r/s are

$$\pm 1, \pm 2, \pm 3, \quad \pm\frac{1}{2}, \pm\frac{3}{2}, \quad \pm\frac{1}{3}, \pm\frac{2}{3}, \quad \pm\frac{1}{6}.$$

Going through the list from left to right and checking individual values (preferably by Ruffini's rule), we find $f(-3/2)$, and so the polynomial has $x + 3/2$ as a factor, or $2x + 3$ if we prefer. Dividing by this factor we find

$$f(x) = 6x^3 + 5x^2 - 10x - 6 = (2x + 3)(3x^2 - 2x - 2).$$

Now the quadratic factor has discriminant $(-2)^2 - 4 \cdot 3 \cdot (-2) = 28$, which is not the square of a rational number. Hence the roots of this factors are irrational, and so this factor cannot be further factorised over $\mathbb{Q}$ (and hence not over $\mathbb{Z}$ either).

Question 7. Let α, β and γ be the complex roots of the polynomial $x^3 - 3x^2 - 2x + 5$. [10 marks]
(The roots are distinct, but you need not check that.)
Compute $\alpha^{-2} + \beta^{-2} + \gamma^{-2}$.

Solution: Because

$$x^3 - 3x^2 - 2x + 5 = (x - \alpha)(x - \beta)(x - \gamma),$$

the coefficients of the polynomial tell us that

$$\begin{aligned} &\alpha + \beta + \gamma = 3 \\ &\alpha\beta + \alpha\gamma + \beta\gamma = -2 \\ &\alpha\beta\gamma = -5. \end{aligned}$$

Now one way to proceed is

$$\begin{aligned} \alpha^{-2} + \beta^{-2} + \gamma^{-2} &= \frac{1}{\gamma^2} + \frac{1}{\beta^2} + \frac{1}{\alpha^2} \\ &= \frac{\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2}{(\alpha\beta\gamma)^2} \\ &= \frac{(\alpha\beta + \alpha\gamma + \beta\gamma)^2 - 2(\alpha\beta\gamma)(\alpha + \beta + \gamma)}{(\alpha\beta\gamma)^2} \\ &= \frac{(-2)^2 - 2 \cdot (-5) \cdot 3}{(-5)^2} = \frac{34}{25} \end{aligned}$$

A different way is to consider the reciprocal polynomial

$$5x^3 - 2x^2 - 3x + 1 = (x - \alpha^{-1})(x - \beta^{-1})(x - \gamma^{-1}),$$

from which we see that

$$\begin{aligned} &\alpha^{-1} + \beta^{-1} + \gamma^{-1} = 2/5 \\ &\alpha^{-1}\beta^{-1} + \alpha^{-1}\gamma^{-1} + \beta^{-1}\gamma^{-1} = -3/5 \\ &\alpha^{-1}\beta^{-1}\gamma^{-1} = -1/5, \end{aligned}$$

and hence

$$\begin{aligned} \alpha^{-2} + \beta^{-2} + \gamma^{-2} &= (\alpha^{-1} + \beta^{-1} + \gamma^{-1})^2 - 2(\alpha^{-1}\beta^{-1} + \alpha^{-1}\gamma^{-1} + \beta^{-1}\gamma^{-1}) \\ &= (2/5)^2 - 2 \cdot (-3/5) = 34/25. \end{aligned}$$

Question 8. Compute 5^{-18} in the finite field $\mathbb{F}_{13}$. [10 marks]

Solution: The inverse of 5 in $\mathbb{F}_{13}$ is -5, because $5 \cdot (-5) = -25 \equiv 1 \pmod{13}$. Hence in $\mathbb{F}_{13}$ we have

$$5^{-18} = -5^{18} = (((-5)^3)^3)^2 = ((5)^3)^2 = (-5)^2 = -1.$$

Alternatively, we may use the fact that $5^{12} = 1$ in $\mathbb{F}_{13}$ according to Fermat's little theorem, and hence

$$5^{-18} = 5^{-6} = 5^6 = (5^3)^2 = (-5)^2 = -1.$$

Both solutions given are instances of a general procedure for computing powers in $\mathbb{F}_{13}$, but if we took into account that $5^2 \equiv -1 \pmod{13}$ as we discovered when finding the inverse of 5, an even faster solution would be $5^{-18} = (5^2)^{-9} \equiv (-1)^{-9} = -1 \pmod{13}$.

Question 9. Express the polynomial $f(x) = x^5 - x^4 + x^3 + 1$ as a product of irreducible polynomials in $\mathbb{F}_5[x]$. [10 marks]

Solution: First we search for linear factors of $f(x)$. Calculation shows that the only roots of $f(x)$ in $\mathbb{F}_5$ are 2 and -2 (which is the same as 3). Dividing $f(x)$ by $(x-2)(x+2) = x^2+1$, we find $f(x) = (x-2)(x+2)(x^3 - x^2 + 1)$.

Now 0, 1, and -1 are not roots of $x^3 - x^2 + 1$, because we have already checked that they are not roots of $f(x)$. However, 2 and -2 might be roots of this cubic polynomial, and so we have to check them. Calculation shows that $2 \in \mathbb{F}_5$ is a root of $x^3 - x^2 + 1$, but -2 is not. Hence

$$f(x) = (x-2)^2(x+2)(x^2+x+2).$$

Now one checks that $2 \in \mathbb{F}_5$ is not a root of $x^2 + x + 2$, hence this polynomial is irreducible (over $\mathbb{F}_5$), and we have found the complete factorisation of $f(x)$ in $\mathbb{F}_5[x]$.

Question 10. Determine the number of monic irreducible polynomials of degree two in $\mathbb{F}_5[x]$. [10 marks]

Solution: There are $5^2 = 25$ distinct polynomials of the form $x^2 + a_1 x + a_0$ with coefficients in $\mathbb{F}_5$. Those which are reducible are either squares $(x-\alpha)^2$, with $\alpha \in \mathbb{F}_5$, or products of the form $(x-\alpha)(x-\beta)$, with $\alpha, \beta \in \mathbb{F}_5$ and $\alpha \neq \beta$. There are 5 of the former type, and $\binom{5}{2} = 5 \cdot 4/2 = 10$ of the latter type. Consequently, the number of monic irreducible quadratic polynomials in $\mathbb{F}_5[x]$ equals $25 - 5 - 10 = 10$.