Ideas of mathematical proof. Assignment

Submissions are to be made via Turnitin by uploading scanned or photographed copies as a single PDF file. Write your name on each sheet. Write on A4-sized paper with blue or black pen. The deadline for submission is Thursday 29 February 3pm.

Note that solutions must be by the methods specified in the problems.

A.1. Use mathematical induction to prove that $5^{2n-1}+1$ is divisible by 6 for any $n \in \mathbb{N}$. [7 marks]

A.2. Let $(b_i)_{i\in\mathbb{N}}$ be a sequence defined recursively as $b_1 = 2$, $b_2 = 4$, and $b_i = 2b_{i-1}+3b_{i-2}-4$ for all $i \geqslant 3$. Use mathematical induction to prove that $b_n = 3^{n-1}+1$ for all $n \in \mathbb{N}$. [7 marks]

A.3. Use mathematical induction to prove that $3^n > 4n^2$ for all positive integers $n \geqslant 4$. [7 marks]

A.4. Let the universal set be $\mathscr{U} = \{x \in \mathbb{Z} \mid -8 \leqslant x \leqslant 8\}$, let A be the set of all even integers in $\mathscr{U}$, let $B = \{x \in \mathscr{U} \mid x^2 < 9\}$, and $C = \{x \in \mathscr{U} \mid x < 0\}$. Determine each of the following sets and list their elements: (1) $A \cup B$; (2) $A \cap \overline{C}$; (3) $A \cap \overline{B}$. [7 marks]

A.5. Use the properties of operations on sets to simplify the expression $\overline{(\overline{A} \cap \overline{B}) \cup B}$. [7 marks]

A.6. Solve the (simultaneous) system of inequalities using intersection of solutions of individual inequalities and write the solution as a union of intervals: [7 marks]

$$\begin{cases} x^2 - 5x + 6 \geqslant 0 \\ x^2 - x > 0. \end{cases}$$

A.7. For each of the following relations R on a given set S, determine if R is transitive, reflexive, symmetric, antisymmetric (in each case giving a proof if yes, or a counterexample if not). Hence state if the relation is an order, or an equivalence, or neither.

(a) $S = \mathbb{R}$ and aRb if $|a| \leqslant |b|$ [7 marks]
(here, $|x|$ denotes the absolute value of a number $x \in R$);

(b) $S = \mathbb{Z}$ and mRn if g.c.d.$(m,n) \neq 1$ [7 marks]
(here, g.c.d.(m,n) is the greatest common divisor).

A.8. Let a relation $\sim$ be defined on $\mathscr{P}(\{a,b,c\})$ by the rule $B \sim C$ if $|B| = |C|$.
(Recall that $\mathscr{P}(\{a,b,c\})$ is the set of all subsets of $\{a,b,c\}$.)

(a) Show that $\sim$ is an equivalence. [6 marks]

(b) Draw the diagram of $\sim$ as a subset of $\mathscr{P}(\{a,b,c\}) \times \mathscr{P}(\{a,b,c\})$. [6 marks]

(c) List all elements of the equivalence class of $\{a,b\}$ with respect to this equivalence $\sim$. [6 marks]

A.9. For each of the following mappings, determine whether it is (1) injective or not, (2) surjective or not (in each case giving a proof if yes, or a counterexample if not).

(a) $f : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$, $f((a,b)) = a - b$. [6 marks]

(b) $f : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$, $f(x) = (2x, 3x)$. [6 marks]

A.10. Let $A = \mathscr{P}(\{u,v,w\})$ be the set of all subsets of $\{u,v,w\}$ and let $f : A \to A$ be a mapping defined by the rule $f(X) = X \cup \{v\}$.

(a) Draw the diagram of the Cartesian product $A \times A$ and indicate f as a subset of $A \times A$. [7 marks]

(b) Determine the image of f and list all elements of this image. [7 marks]