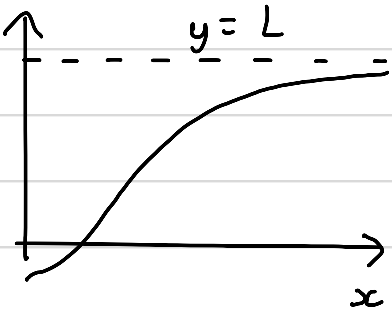
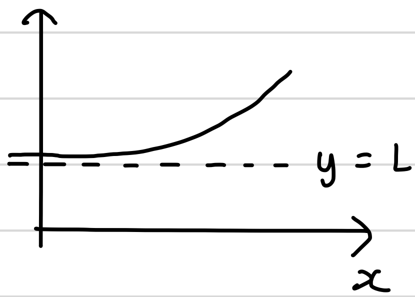
Asymptotes

Definition Suppose that either $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$. Then, the line $y = L$ is a horizontal asymptote to the graph of $y = f(x)$.



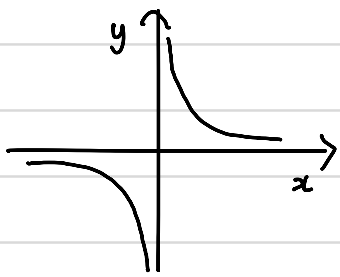
$\lim_{x \to \infty} f(x) = L$



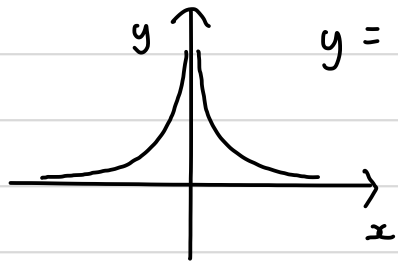
$\lim_{x \to -\infty} f(x) = L$

Definition The line $x = a$ is a vertical asymptote to the graph of $y = f(x)$ if at least one of $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ is equal to ∞ or $-\infty$

Examples

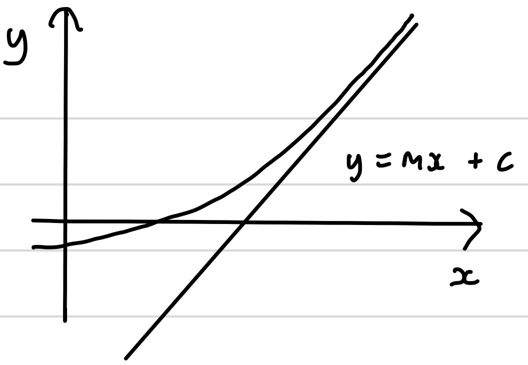


$y = \frac{1}{x}$



$y = 1/x^2$

Definition If $\lim_{x \to \infty} [f(x) - (mx + c)] = 0$, for $m \neq 0$, then the line $y = mx + c$ is a slant asymptote of the curve.



This occurs for rational functions when the degree of the numerator is one higher than the degree of the denominator.

Checklist for curve sketching.

1. Domain
2. Intercepts with axes
3. Symmetry and periodicity
4. Asymptotes : vertical, horizontal, slant
5. Intervals of increase and decrease
6. Local maxima and minima
7. Concavity and points of inflection.

Example Sketch the curve $y = \dfrac{2x^2}{x^2 - 1}$

(1) Domain : $\{x \mid x^2 - 1 \neq 0\} = \{x \mid x \neq \pm 1\}$

$= (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

(2) Intercepts : $y = 0$ when $x = 0$.

(3) $f(-x) = \dfrac{2(-x)^2}{(-x)^2 - 1} = \dfrac{2x^2}{x^2 - 1} = f(x)$

and the function is even.

(4) $$\lim_{x \to \pm\infty} \frac{2x^2}{x^2 - 1} = \lim_{x \to \pm\infty} \frac{2}{1 - 1/x^2} = 2$$

and $y = 2$ is a (double) horizontal asymptote.

The denominator is 0 when $x = \pm 1$, so we have that

$$\lim_{x \to 1^+} \frac{2x^2}{x^2 - 1} = \infty, \quad \lim_{x \to 1^-} \frac{2x^2}{x^2 - 1} = -\infty$$

$$\lim_{x \to -1^+} \frac{2x^2}{x^2 - 1} = -\infty, \quad \lim_{x \to -1^-} \frac{2x^2}{x^2 - 1} = \infty$$

and we have vertical asymptotes at $x = \pm 1$.

(5) $$f'(x) = \frac{(x^2 - 1)(4x) - 2x^2 . 2x}{(x^2 - 1)^2}$$
$$= \frac{-4x}{(x^2 - 1)^2}$$

So, $f'(x) > 0$ when $x < 0$ (with $x \neq -1$)
and $f'(x) < 0$ when $x > 0$ (with $x \neq 1$).
Then $f(x)$ is increasing on $(-\infty, -1)$ and $(-1, 0)$ and decreasing on $(0, 1)$ and $(1, \infty)$.

(6) $f' = 0$ at $x = 0$ and does not exist at $x = \pm 1$. We have already dealt with the vertical asymptotes at $x = \pm 1$. We know that $(0, 0)$ is a local maximum, since $f'(x)$ changes from positive to negative here.

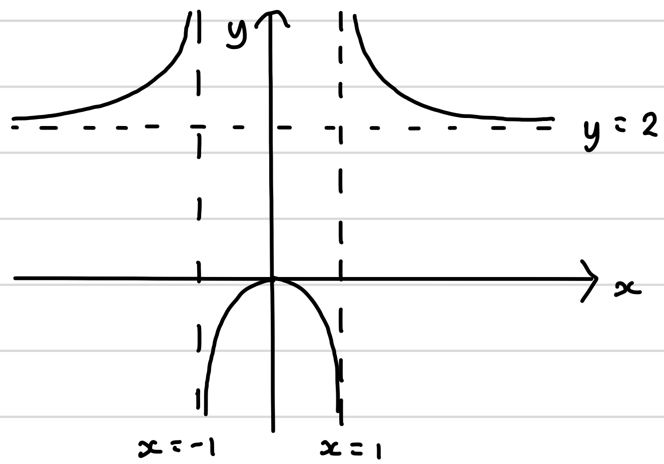
(7) $$f''(x) = \frac{(x^2-1)^2(-4) + 4x.2(x^2-1)2x}{(x^2-1)^4}$$

$$= \frac{-4(x^2-1) + 16x^2}{(x^2-1)^3}$$

$$= \frac{12x^2 + 4}{(x^2-1)^3}$$

Since $12x^2 + 4 > 0$ for all x, we have that $f''(x) > 0$ when $x^2 > 1$ and that $f''(x) < 0$ when $x^2 < 1$. This means that the curve is concave upwards on $(-\infty, -1)$ and $(1, \infty)$ and concave downwards on $(-1, 1)$.

Putting all this together, we find that



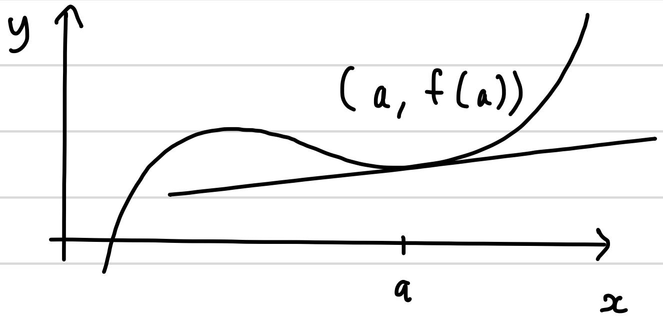
Taylor polynomials and Taylor series

- We can use polynomials to approximate complicated functions.
- A systematic way of doing this is to use Taylor polynomials.
- We build up to this by looking at linear and quadratic approximations.

Linearisation

- Imagine we have a function, $f(x)$, that can be differentiated as many times as we like.
- Near the point $(a, f(a))$, we can approximate the function by its tangent line,

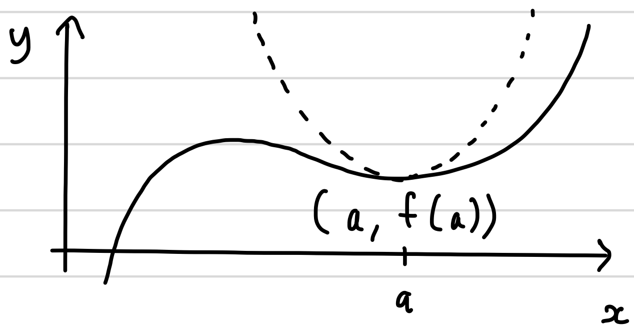
$$y = f(a) + f'(a)(x-a)$$



- The closer we get to $x = a$, the better the approximation becomes.

Quadratic approximations

- It is often possible to improve the approximation by using a quadratic.



- This is given by

$$y = P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

Why does this work?

① When $x = a$, $P_2(a) = f(a)$, so P_2 and the function match at $x = a$.

② $P_2'(x) = f'(a) + f''(a)(x-a)$

Putting $x = a$, we see that $P_2'(a) = f'(a)$, and the derivatives match at $x = a$.

③ $P_2''(x) = f''(a)$, so the second derivatives match at $x = a$.

- Our approximating polynomial therefore reproduces some important features of $f(x)$.

- Note that $P_2'''(x) = 0$ for all x, so $P_2(x)$ contains no information about the third (and higher) derivatives of $f(x)$.

Higher-degree approximations

We can improve the approximation further by including more terms:

$$P_N(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(N)}(a)}{N!}(x-a)^N$$

or, in sigma notation,

$$P_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

- This is the N^{th} order Taylor polynomial of $f(x)$ at $x = a$
- All derivatives of $P_N(x)$, up to and including the N^{th} derivative, match those of $f(x)$, that is,

$$P_N^{(n)}(a) = f^{(n)}(a) \quad \text{for } n = 0, 1, 2, \ldots, N$$

but all higher derivatives of P_N must be zero everywhere.

- The function P_N includes the information about f that comes from its derivatives up to order N at $x = a$.

<u>Example</u>: Find the third-order Taylor polynomial approximation to $f(x) = e^x$, valid near $a = 0$.

$$P_3(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f^{(3)}(0)x^3}{3!}$$

All derivatives of e^x with respect to x are e^x, so

$$f(0) = f'(0) = f''(0) = f^{(3)}(0) = e^0 = 1$$

We then have that $P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$

<u>The error term</u>

- We would now like to quantify the error made when approximating a function by a Taylor polynomial.
- To begin, we define the N^{th}-order error term as

$$R_N(x) = f(x) - P_N(x)$$

Taylor's theorem then states that the N^{th} order error term about $x = a$ is

$$R_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}(x-a)^{N+1}$$

where c is a number that lies between x and a.

__Note__

- Taylor's theorem is an extension of the mean value theorem, and is exactly the same if we set $N = 0$ in the above formula. In this case, $R_0(x) = f'(c)(x-a)$, so that $f(x) - f(a) = f'(c)(x-a)$. Rearranging this, we find that

$$f'(c) = \frac{f(x) - f(a)}{x - a}$$

for some c between x and a, which is the mean value theorem.

__Example__ We found earlier that the third-order Taylor approximation to e^x around $a = 0$ is

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

Taylor's theorem states that the error term is given by

$$R_3(x) = \frac{f^{(4)}(c)\,x^4}{4!}$$

where c is between 0 and x. Again, any derivative of e^x with respect to x is e^x, and $4! = 24$, so,

$$R_3(x) = \frac{e^c x^4}{24}$$

Taylor's theorem does not give us the value of c. However, we can calculate the maximum error for a given value of x. For example, if we use the polynomial given above to estimate $e^{-1/10}$ by expanding about $a = 0$, we find that

$$e^{-1/10} = 1 - \left(\frac{1}{10}\right) + \frac{(1/100)}{2} - \frac{(1/1000)}{6} + \frac{e^c}{24}\left(\frac{1}{10000}\right)$$

$$= \frac{5429}{6000} + \frac{e^c}{240000}$$

We know that $-1/10 < c < 0$, so the maximum error term is $\frac{e^0}{240000} = \frac{1}{240000}$.

Note: the key when using the above technique to estimate the value of a function at a given point is to start at a nearby point where the value of $f(x)$ is known. In the above example, we knew that $e^0 = 1$.

Taylor series

- If the number of terms becomes infinite ($N \to \infty$), a Taylor polynomial becomes an infinite series.
- This is a convergent series: it approaches a limit.

Example: the Taylor series about $x = 1$ of e^x is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)(x-1)^n}{n!} = \sum_{n=0}^{\infty} \frac{e^1(x-1)^n}{n!} = e \sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}$$

since all the derivatives $f^{(n)}(x)$ of e^x are e^x.

Example (Exam 2021/22)

Find the first three non-zero terms in the Taylor series of $\cos x$ around $x = \pi/2$.

n	$f^{(n)}(x)$	$f^{(n)}(\pi/2)$
0	$\cos x$	0
1	$-\sin x$	-1
2	$-\cos x$	0
3	$\sin x$	1
4	$\cos x$	0
5	$-\sin x$	-1

$$f(x) = f(\pi/2) + f'(\pi/2)(x-\pi/2)$$

$$+ \frac{f''(\pi/2)}{2!}(x-\pi/2)^2 + \frac{f^{(3)}(\pi/2)}{3!}(x-\pi/2)^3$$

$$+ \frac{f^{(4)}(\pi/2)}{4!}(x-\pi/2)^4 + \frac{f^{(5)}(\pi/2)}{5!}(x-\pi/2)^5$$

$$+ \cdots$$

$$= -(x-\pi/2) + \frac{(x-\pi/2)^3}{3!} - \frac{(x-\pi/2)^5}{5!} + \cdots$$

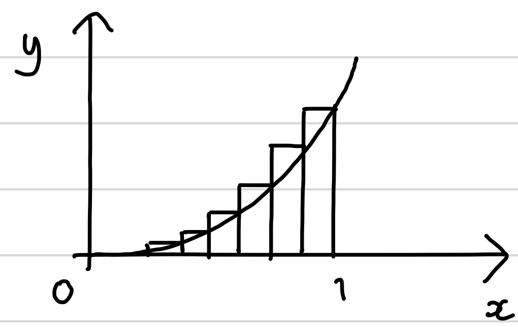
Integration

Integration from first principles

Suppose that a function is continuous and non-negative on an interval $x \in I$, and we wish to determine the area under the curve on I.

Example: What is the area under the curve $y = x^2$ on the interval $x \in [0, 1]$?

We can estimate the area using rectangles:



- Our estimate becomes more accurate as we increase the number of rectangles.
- Suppose that we have n rectangles. Since the interval is of width 1, each rectangle is of width $1/n$.

- The height of the first rectangle is $\left(\frac{1}{n}\right)^2$, that of the second is $\left(\frac{2}{n}\right)^2$, and so on.
- The sum of the areas of the rectangles is given by

$$R_n = \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \cdots + \frac{1}{n}\left(\frac{n}{n}\right)^2$$
$$= \frac{1}{n^3}\left(1^2 + 2^2 + \cdots + n^2\right)$$

- To evaluate this sum, we need to know that the sum of the squares of the first n positive integers is given by

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

- Taking the limit of an infinite number of rectangles, we find that

$$\lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3} = \lim_{n\to\infty} \frac{2 + 3/n + 1/n^2}{6} = \frac{1}{3}$$

<u>Definition</u> Suppose that $f(x)$ is a continuous function on the interval $[a, b]$. We divide the interval into n sub-intervals of equal width $\Delta x = (b-a)/n$. Let $x_0 = a$, $x_1, x_2, \dots, x_n = b$ denote the endpoints of the subintervals, and let $c_i \in [x_{i-1}, x_i]$ be sample points. Then the definite integral of f from a to b is

$$\int_a^b f(x)\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i)\, \Delta x$$

<u>Notes</u>

- if the limit exists, we say that the function is integrable
- the limit gives the same value for all possible choices of sample points c_i.
- The sum $\sum_{i=1}^{n} f(c_i)\, \Delta x$ is called the Riemann sum
- Since the value of the limit is independent of the sample of points chosen, we can simplify the definition of the definite integral to

$$\int_a^b f(x)\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\, \Delta x$$

where $x_i = a + i\,\Delta x$.

Properties of the definite integral

① $\int_a^b c \, dx = c(b-a)$, where c is any constant

② $\int_a^b [f(x) \pm g(x)] \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx$

③ $\int_a^b c \, f(x) \, dx = c \int_a^b f(x) \, dx$

④ $\int_a^c f(x) \, dx + \int_c^b f(x) \, dx = \int_a^b f(x) \, dx$

(splitting the range)

⑤ If $f(x) \geqslant 0$ for $a \leqslant x \leqslant b$, then

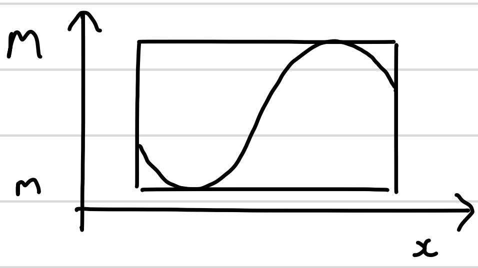
$$\int_a^b f(x) \, dx \geqslant 0$$

⑥ If $f(x) \geqslant g(x)$ for $a \leqslant x \leqslant b$, then

$$\int_a^b f(x) \, dx \geqslant \int_a^b g(x) \, dx$$

⑦ If $m \leqslant f(x) \leqslant M$ for $a \leqslant x \leqslant b$, then

$$m(b-a) \leqslant \int_a^b f(x) \, dx \leqslant M(b-a)$$



__Note__ : this may be used to estimate the definite integrals of complicated functions

Fundamental theorem of calculus

- This essentially states that integration and differentiation are the inverses of each other.

Part ① If f is continuous on $[a, b]$, then the function F defined by

$$F(x) = \int_a^x f(t)\,dt, \qquad a \leq x \leq b$$

is continuous on $[a, b]$, differentiable on (a, b) and $\underline{F'(x) = f(x)}$.

Alternatively, $$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Notes: - the choice of the lower limit is not important: it sets the constant of integration, which vanishes when we differentiate.
- this theorem allows us to view integration as solving a differential equation: we are given $F'(x) = f(x)$, and then find $F(x)$.

Antiderivatives

If F is a function whose derivative is f, so that $F'(x) = f(x)$, we say that F is an antiderivative of f.

<u>Example</u>: Any function $\frac{x^3}{3} + c$, for some constant c, is an antiderivative of x^2.

<u>Fundamental theorem of calculus part 2</u>

If f is continuous on $[a, b]$, then,

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f.

<u>Example</u>: $\int_0^1 x^2\,dx = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$

<u>Indefinite integrals</u>

- The indefinite integral $\int f(x)\,dx$ is the family of all antiderivatives of f.
  <u>Example</u>
  $$\int x^2\,dx = \frac{x^3}{3} + c$$
- We can find many indefinite integrals $\int f(x)\,dx$ by recalling the function $F(x)$ such that $F'(x) = f(x)$ e.g.
  $\int \cos x\,dx = \sin x + c.$