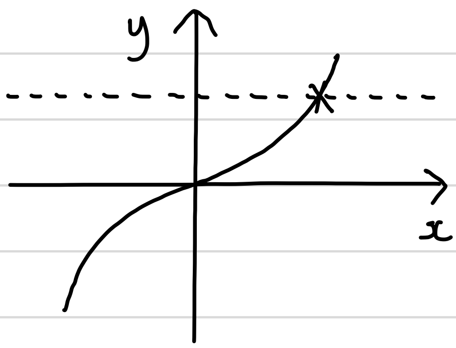
Inverse functions

- A function $f(x)$ takes an input, x, to give an output, y.
- Its inverse, $f^{-1}(y)$, tells us the value of x we should input to the function to get a given output, y.
- We can only define the inverse of one-to-one functions, which never give the same output for different values of x; that is, $f(x_1) \neq f(x_2)$ when $x_1 \neq x_2$.

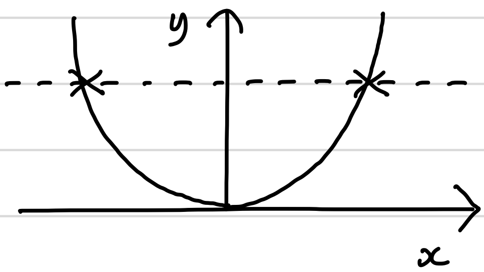
- Any horizontal line drawn through the graph of a one-to-one function will intercept it only once.

Examples



$f(x) = x^3$
$x \in \mathbb{R}$

one-to-one



$f(x) = x^2$
$x \in \mathbb{R}$, not one-to-one.

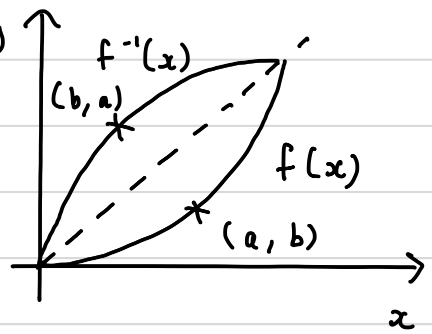
Definition Let the function f be one-to-one with domain X and range Y. Then, its inverse function has domain Y and range X

and is defined by

$$f^{-1}(y) = x \quad \text{for any } y \in Y.$$

Graph of the inverse

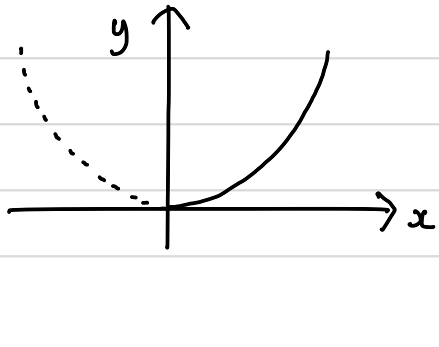
- If $f(a) = b$, then $f^{-1}(b) = a$.
- This means that, if (a, b) is on the graph of $f(x)$, then (b, a) is on the graph of $f^{-1}(x)$.



- Then, the graph of f^{-1} is found by reflecting the graph of f about $y = x$.

- Functions that are not one-to-one can be made so by <u>restricting their domain</u>.

e.g. $f(x) = x^2$ can be made one-to-one by restricting its domain to $[0, \infty)$.



Calculating the inverse

① Write $y = f(x)$.

② Solve this equation for x in terms of y.

③ If required, interchange x and y to express f^{-1} as a function of x: $y = f^{-1}(x)$.

Inverse hyperbolic functions

Given that $\sinh x = \dfrac{e^{x} - e^{-x}}{2}$, find $\sinh^{-1} x$.

Write $y = \sinh x$

$$y = \frac{e^{x} - e^{-x}}{2}$$

$$2y = e^{x} - e^{-x}$$

and $e^{x} - 2y - e^{-x} = 0$

so $e^{2x} - 2ye^{x} - 1 = 0$

This is a quadratic in e^{x}, with solution

$$e^{x} = \frac{2y \pm \sqrt{4y^2 + 4}}{2} = y \pm \sqrt{y^2 + 1}$$

so that $x = \ln\left(y \pm \sqrt{y^2 + 1}\right)$.

Since $\sqrt{y^2+1} > y$, and we cannot take the logarithm of a negative number, we must take the positive root, so that

$$x = \ln\left(y + \sqrt{y^2+1}\right)$$

and $\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$

Maximum and minimum values of functions

- We distinguish absolute, or global, maxima and minima from local maxima and minima.

Definition Let c be a number in the domain D of a function f. Then, $f(c)$ is the

- absolute maximum value of f on D if $f(c) \geq f(x)$ for all x in D.
- absolute minimum value of f on D if $f(c) \leq f(x)$ for all x in D.

- Local maxima and minima are defined on small intervals rather than the whole domain.

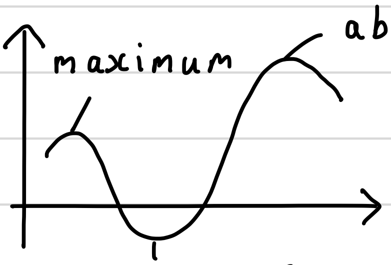
Definition The number $f(c)$ is a

- local maximum value of f if $f(c) \geq f(x)$ when x is near c.

- <u>local minimum value</u> of f if $f(c) \le f(x)$ when x is near c.

<u>Note</u> : every absolute maximum (or minimum) also satisfies the definition of a local maximum (or minimum).

<u>Example</u>

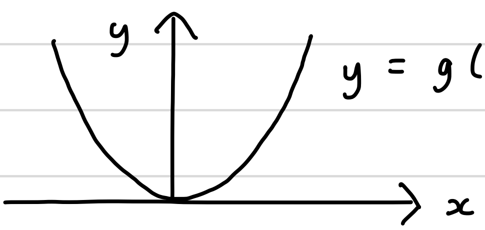


<u>Further examples</u>

(1) $f(x) = \sin x$ takes on its absolute (and local) maximum and minimum values infinitely many times, at intervals of 2π.

(2) $g(x) = x^2$

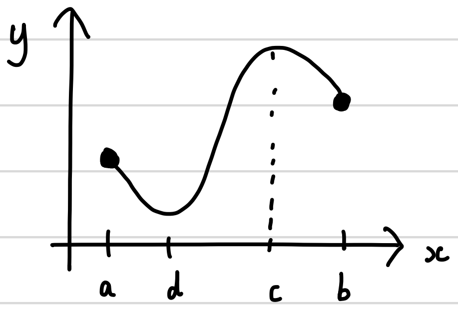
$g(0)$ is the absolute (and local) minimum value. There is <u>no</u> absolute maximum.

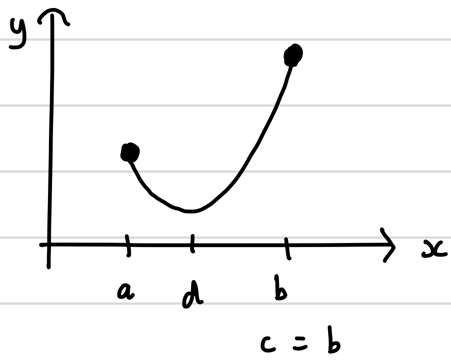


<u>Note</u>: maximum and minimum values are often called <u>extrema</u>, or <u>extreme values</u>.

The extreme value theorem

If $f(x)$ is continuous on a closed interval $[a, b]$, then it has an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.



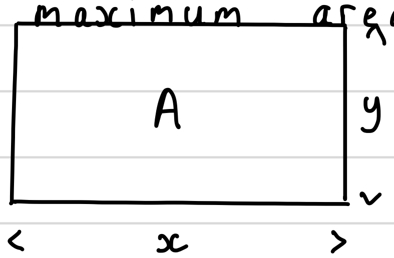


- Extreme values can occur at the endpoints of the interval, as well as at peaks and troughs.

Fermat's theorem

If $f(x)$ has a local extremum (i.e. maximum or minimum value) at c, and $f'(c)$ exists, then $f'(c) = 0$.

Example Show that, of all rectangles with the same perimeter, the square has maximum area.



Area $A = xy$

Perimeter $p = 2x + 2y$

The perimeter is fixed

to a value p: this is called a <u>constraint</u>. We want to find the maximum value of A for a given p.

- Firstly, we use the equation for p to write y in terms of x: $y = \frac{p - 2x}{2}$

- We then substitute this in the expression for A to find that

$$A = x\left(\frac{p-2x}{2}\right) = \frac{px}{2} - x^2$$

- We now find dA/dx and set it to 0, so that $\frac{p}{2} - 2x = 0$, and

$$x = \frac{p}{4}, \quad y = \frac{p - 2(p/4)}{2} = \frac{p}{4}$$

and we have a square.

- We can confirm that this is a maximum by noting that $\frac{d^2A}{dx^2} = -2 < 0$

- This is an example of an <u>optimisation problem</u> with a constraint.

To find the absolute maximum and minimum values of a continuous function on a closed interval $[a, b]$, we

(1) Find the values of f at the stationary points.

(2) Find the values of f at a and b.

(3) The absolute maximum and minimum values are the largest and smallest values respectively from steps (1) and (2).

<u>Example</u> Find the absolute maximum and minimum values of $f(x) = 3x^2 - 12x + 5$ on $[0, 3]$, giving a justification of your answer.

- The function is continuous on the given interval, and its absolute maximum and minimum will each occur at a stationary point or an endpoint of the interval.

(1) $f'(x) = 6x - 12$
$f'(x) = 0$ at $x = 2$, and
$f(2) = 3 \times 2^2 - 12 \times 2 + 5 = -7$

(2) At the endpoints, we have that $f(0) = 5$ and $f(3) = 3 \times 3^2 - 12 \times 3 + 5 = -4$.

(3) The absolute maximum is 5 and the absolute minimum is -7.

<u>Critical points</u>

<u>Definition</u> A critical point of a function

is any value c in its domain where $f'(c) = 0$ or $f'(c)$ does not exist.

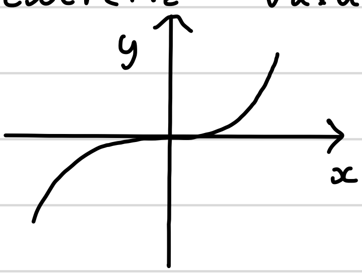
Example : Find the critical points of $f(x) = x^{2/3}(1-x)$. We have that $f(x) = x^{2/3} - x^{5/3}$, so that

$$f'(x) = \frac{2}{3}x^{-1/3} - \frac{5}{3}x^{2/3} = \frac{2-5x}{3x^{1/3}}$$

$f'(x)$ is 0 at $x = 2/5$ and does not exist at $x = 0$. These are the two critical points of $f(x)$.

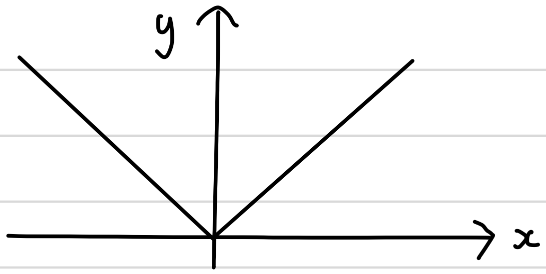
<u>Notes</u>

- If $f'(c) = 0$, there is not necessarily a maximum or minimum value of f at c.
  <u>Example</u> : If $f(x) = x^3$, then $f'(0) = 0$, but there is no extreme value.



- There may be an extremum even when $f'(x)$ does not exist

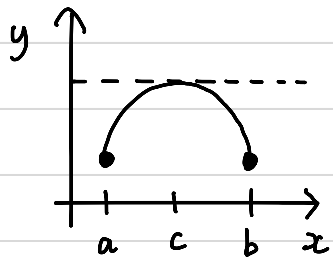
Example



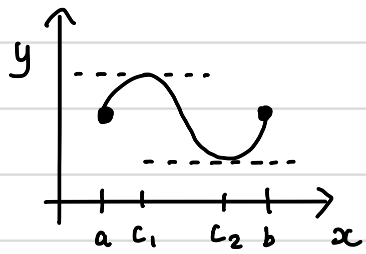
$f(x) = |x|$

Rolle's theorem

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$ then there must be at least one number c in (a, b) such that $f'(c) = 0$.



1



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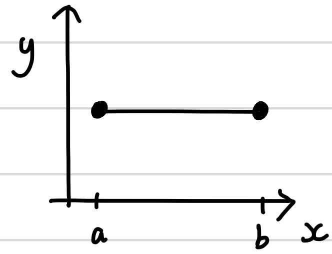


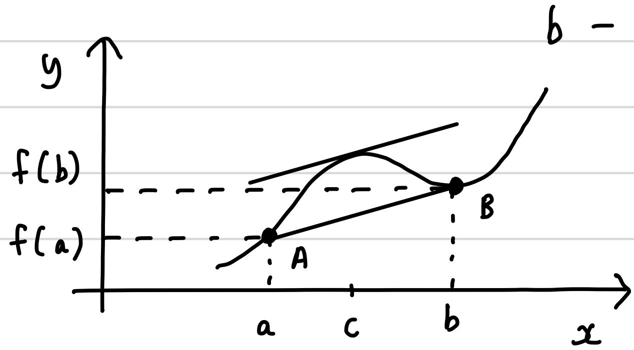
3

Case 1 : one point where $f' = 0$.
" 2 : two points where $f' = 0$.
" 3 : $f' = 0$ everywhere in the interval.

Mean value theorem

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b). Then, there exists at least one number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



$y = f(x)$

To understand the mean value theorem, note that the equation of the line AB is

$$g(x) = f(a) + (x-a)\frac{f(b)-f(a)}{b-a}$$

The difference between the curve and the line is

$$h(x) = f(x) - g(x) = f(x) - f(a) - (x-a)\frac{f(b)-f(a)}{b-a}$$

Since the curve and the line intersect at A and B, $h(x) = 0$ at both of these points. We can then apply Rolle's theorem, which tells us that $h'(x) = 0$ for at least one point c between A and B. Differentiating $h(x)$, we find that

$$h'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}.$$

Since $h'(c) = 0$, we have that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$: the mean value theorem

Intervals of increase and decrease

<u>Definition</u> Suppose that f is continuous on an interval I. Then, f is an increasing

function if, for every pair $x_1, x_2 \in I$, where $x_1 < x_2$, we have that $f(x_1) < f(x_2)$. Conversely, f is a decreasing function if, for every pair $x_1, x_2 \in I$, where $x_1 < x_2$, we have that $f(x_1) > f(x_2)$.

Then, for differentiable functions,

1. If $f'(x) > 0$ for all $x \in I$, then f is increasing on the interval I.
2. If $f'(x) < 0$ for all $x \in I$, then f is decreasing on the interval I.

Proof of 1 (2 is proved similarly)

The function is differentiable on (x_1, x_2), so we can apply the mean value theorem, which tells us that there is a number $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

We know that $f'(c) > 0$ and that $x_2 - x_1 > 0$. It then follows that $f(x_2) - f(x_1) > 0$, so that $f(x_2) > f(x_1)$.

Since this argument can be applied to any pair $x_1, x_2 \in I : x_1 < x_2$, we have shown that f is increasing on the interval I.

To find the intervals of increase or decrease for a function $f(x)$ on its domain D, we

1. Find the critical points of f (and other points where f' does not exist, e.g, $x = 0$ for $f(x) = 1/x$)
2. Divide D into sub-intervals with endpoints at these points.
3. Check the sign of f' within each subinterval.
4. If $f' > 0$, then f is increasing on that interval. If $f' < 0$, then f is decreasing.

<u>Example</u> Determine the intervals of increase and decrease for $f(x) = x^3 - 3x^2 + 2$, $x \in \mathbb{R}$.

1. Find the critical points of f
$f'(x) = 3x^2 - 6x = 3x(x-2)$
This exists for all $x \in \mathbb{R}$, and $f'(x) = 0$ at $x = 0$ and $x = 2$.

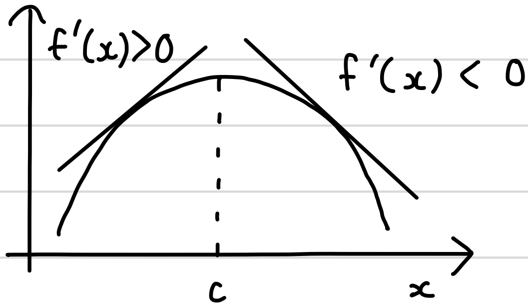
2. We divide the domain, $\mathbb{R}$, into three subintervals: $(-\infty, 0)$, $(0, 2)$ and $(2, \infty)$.
3. We now choose a point in each interval, and check the sign of f' at that point.
$f'(-1) = 9 > 0$, $f'(1) = -3 < 0$ and $f'(3) = 9 > 0$.
4. So, f is increasing on $(-\infty, 0)$
decreasing on $(0, 2)$
and increasing on $(2, \infty)$.

Classifying critical points as local maxima or minima

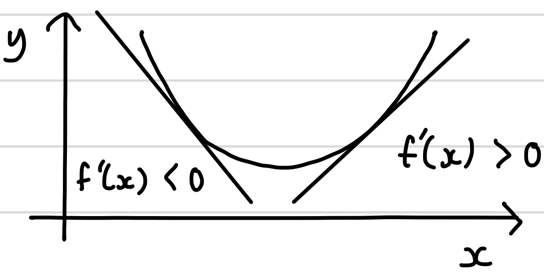
Using the first derivative

Suppose that $f'(c) = 0$ for some $c \in (a, b)$. Then,

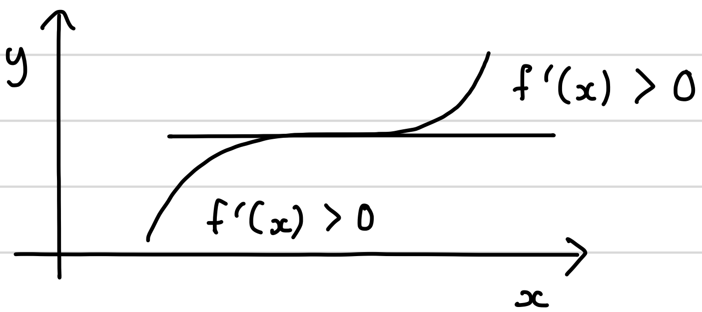
1. if $f'(x) > 0$ on (a, c) and $f'(x) < 0$ on (c, b), then f has a local maximum at $x = c$.



2. if $f'(x) < 0$ on (a, c) and $f'(x) > 0$ on (c, b), then f has a local minimum at $x = c$.



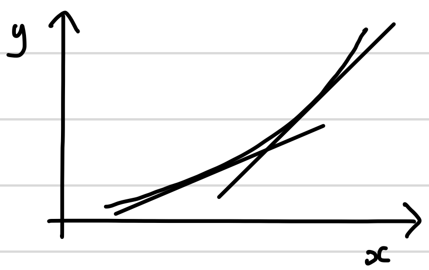
3. if $f'(x) > 0$ on (a, c) and $f'(x) > 0$ on (c, b), or $f'(x) < 0$ on (a, c) and $f'(x) < 0$ on (c, b), then f has neither a local maximum nor a local minimum at $x = c$.

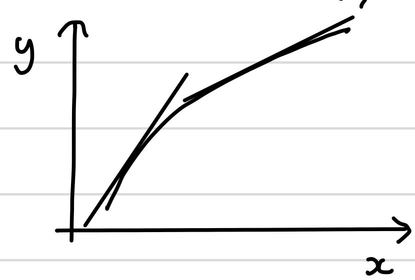


Concavity

Definition Suppose that the function f is differentiable on $x \in (a, b)$. Then,

1. if $f'(x)$ is increasing on (a, b), the graph is concave upwards on (a, b).
2. if $f'(x)$ is decreasing on (a, b), the graph is concave downwards on (a, b).





Concave upwards: curve lies above its tangent at all points in the interval

Concave downwards: curve below tangent.

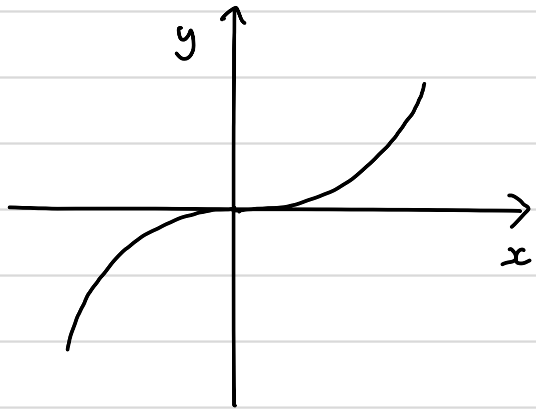
Testing for concavity

Suppose that the function $f(x)$ is twice differentiable on $x \in (a, b)$. Then,

1. if $f''(x) > 0$ for all $x \in (a, b)$, then the graph of f is concave upwards on (a, b).

② if $f''(x) < 0$ for all $x \in (a, b)$, then the graph of f is concave downwards on (a, b).

Example



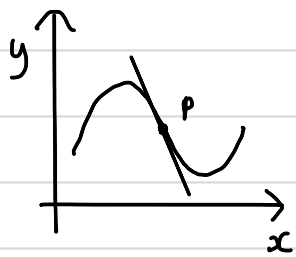
If $f(x) = x^3$, then $f'(x) = 3x^2$
and $f''(x) = 6x$

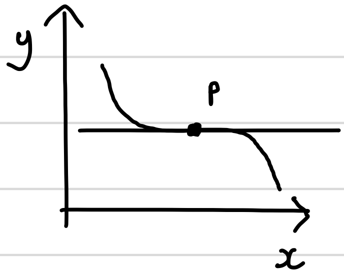
- For $x < 0$, $f''(x) < 0$ and f is concave downwards.
- For $x > 0$, $f''(x) > 0$ and f is concave upwards.

Points of inflection

Definition: Suppose that the function $f(x)$ is continuous on an interval I. If there

is a point $P \in I$ where the graph changes from concave upwards to concave downwards (or vice-versa), then P is an inflection point of f.





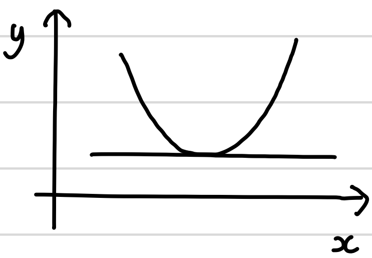
horizontal tangent

At a point of inflection, the curve crosses its tangent.

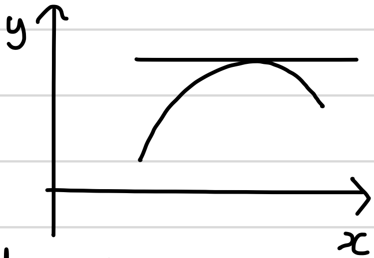
Second derivative test for maxima and minima

Suppose that the function f is continuous near $x = c$.
Then,

1. if $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
2. if $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.



$f'(c) = 0$
$f''(c) > 0$



$f'(c) = 0$
$f''(c) < 0$

local min, concave up

local max, concave down

Note: this test fails when $f''(c) = 0$, or does not exist.

Example: $f(x) = x^4$ has a local minimum at $x = 0$, but $f''(x) = 0$ here.