Continuity and differentiability.

Theorem If a function is differentiable at x_0, then it is continuous at x_0.

To prove continuity at x_0, we need to show that $\lim_{x \to x_0} f(x) = f(x_0)$, i.e. that

$$\lim_{h \to 0} f(x_0 + h) = f(x_0)$$

Proof Suppose that the function $f(x)$ is differentiable on an open interval containing the point x_0. We then know that the derivative

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists. We now consider the new limit

$$\lim_{h \to 0} \left[\frac{f(x_0 + h) - f(x_0)}{h} \right] \times h$$

and write it in two ways.

Firstly, we write $\lim_{h \to 0} \left[\frac{f(x_0 + h) - f(x_0)}{h} \right] \times \lim_{h \to 0} h$

(by the product rule for limits)

$$= f'(x_0) \lim_{h \to 0} h = 0$$

since we are given that $f'(x_0)$ exists (and does not diverge, become indeterminate etc.)

Secondly, $\lim_{h \to 0} \left[\frac{f(x_0 + h) - f(x_0)}{h} \right] h$

$$= \lim_{h \to 0} [f(x_0 + h) - f(x_0)]$$

Equating the two results:

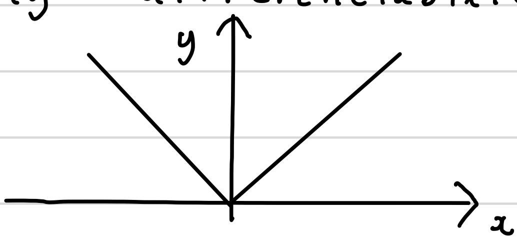
$$\lim_{h \to 0} [f(x_0 + h) - f(x_0)] = 0$$

and $\lim_{h \to 0} f(x_0 + h) = f(x_0)$

so that the function is continuous at x_0.

Continuous non-differentiable functions

- Differentiability implies continuity, but continuity does not imply differentiability.



$y = f(x)$
$= |x|$

The absolute value function is continuous at $x = 0$. However, if we try to evaluate its derivative here, we find that

$$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1$$

and

$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1$$

Since the left- and right-sided limits are different, $\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$ does not exist,

and $f(x)$ is not differentiable at $x = 0$.

Rules for differentiation

Suppose that $f(x)$ and $g(x)$ are differentiable functions on an open domain. Then,

1. $\frac{d}{dx}[c f(x)] = c f'(x)$, where c is a constant

2. $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$

3. $\frac{d}{dx}[f(x) g(x)] = f'(x) g(x) + f(x) g'(x)$

   (the product rule)

roof $\frac{d}{dx}[f(x) g(x)] = \lim_{h \to 0} \frac{f(x+h) g(x+h) - f(x) g(x)}{h}$

$$= \lim_{h \to 0} \frac{f(x+h)\,g(x+h) - f(x)\,g(x+h) + f(x)\,g(x+h) - f(x)\,g(x)}{h}$$

$$= \lim_{h \to 0} \frac{[f(x+h) - f(x)]}{h} g(x+h)$$

$$+ \lim_{h \to 0} f(x) \frac{[g(x+h) - g(x)]}{h}$$

$$= f'(x)\,g(x) + f(x)\,g'(x)$$

Example If $y = xe^x$, then

$$\frac{dy}{dx} = \frac{d}{dx}(x)\,e^x + x\frac{d}{dx}e^x$$

$$= e^x + xe^x$$

4 $$\frac{d}{dx}[f(g(x))] = f'(g(x))\,g'(x)$$

Proof (not valid for all functions)

$$\frac{d}{dx}[f(g(x))] = \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$

$$= \lim_{h \to 0} \left[\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \times \frac{g(x+h) - g(x)}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \right] \times \lim_{h \to 0} \left[\frac{g(x+h) - g(x)}{h} \right]$$

(product rule for limits)

$= \quad f'(g(x))\, g'(x)$

(for a fuller explanation, see J. Stewart, Calculus)

<u>Alternative notation</u>: if y is a function of u and u is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

<u>Example</u> Differentiate $\frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$

(the normal distribution)

Here, $u(x) = -x^2/2$ and $g(u) = \frac{1}{\sqrt{2\pi}} \exp(u)$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{\sqrt{2\pi}} e^{u} \cdot (-x) = \frac{-x}{\sqrt{2\pi}} e^{-x^2/2}$$

(5) $$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{[g(x)]^2} \quad \text{for } g(x) \neq 0$$

(the quotient rule)

<u>Example</u>: Differentiate $y = \frac{x^2 + x - 2}{x^3 + 6}$

$$\frac{dy}{dx} = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{[g(x)]^2}$$

$$= \frac{(2x+1)(x^3+6) - (x^2+x-2)(3x^2)}{(x^3+6)^2}$$

Logarithmic differentiation

- This is a technique for differentiating complicated products or quotients of functions
- It is related to implicit differentiation.

- Recall: to find dy/dx when (for example) $x^2 + y^2 = 25$, we differentiate both sides to find

$$2x + 2y\frac{dy}{dx} = 0$$

(where we have used the chain rule on the second term with y as the inner function) and $\frac{dy}{dx} = \frac{-x}{y}$

- In logarithmic differentiation, we begin by writing $y = f(x)$, where $f(x)$ is the function to be differentiated. We then
  1. Take natural logarithms of both sides, and simplify $\ln(f(x))$ using the laws of logarithms:

(i) $\ln(ab) = \ln a + \ln b$
(ii) $\ln(a/b) = \ln a - \ln b$
(iii) $\ln(a^r) = r \ln a$

2. Differentiate both sides of the equation with respect to x.
3. Solve the resulting equation for dy/dx.

Example Differentiate $f(x) = \dfrac{x^2 \sin x}{\cos 2x}$

Write $y = \dfrac{x^2 \sin x}{\cos 2x}$, and

1. Take logarithms of both sides:

$$\begin{aligned}\ln y &= \ln(x^2) + \ln(\sin x) - \ln(\cos 2x) \\ &= 2 \ln x + \ln(\sin x) - \ln(\cos 2x)\end{aligned}$$

2. Differentiate both sides with respect to x:

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + \frac{\cos x}{\sin x} - \frac{1}{\cos 2x}(-2 \sin 2x)$$

3. Solve for dy/dx (and simplify):

$$\begin{aligned}\frac{dy}{dx} &= y\left[\frac{2}{x} + \cot x + 2 \tan 2x\right] \\ &= \frac{x^2 \sin x}{\cos 2x}\left[\frac{2}{x} + \cot x + 2 \tan 2x\right]\end{aligned}$$

where we have remembered the original formula for y.

Notation for higher-order derivatives

- We tend to use a number in brackets rather than repeated primes:

$$\text{e.g.} \quad \frac{d^4 y}{dx^4} = f^{(4)}(x) = f^{(iv)}(x)$$