Evaluating limits

Direct substitution

Example Evaluate $\lim_{x \to -1} \dfrac{x^2 - 3x + 2}{x - 2}$

Substituting $x = -1$ into the above expression yields $\dfrac{6}{-3} = -2$.

Indeterminate forms

What happens if we try to calculate

$\lim_{x \to 2} \dfrac{x^2 - 3x + 2}{x - 2}$ by direct substitution?

We get $\dfrac{4 - 6 + 2}{2 - 2} = \dfrac{0}{0}$: an indeterminate form

However, we can factorise the numerator to get

$$\lim_{x \to 2} \frac{x^2 - 3x + 2}{x - 2} = \lim_{x \to 2} \frac{(x-2)(x-1)}{x - 2}$$

$$= \lim_{x \to 2} x - 1 = 1$$

Note: the simplification $\dfrac{x^2 - 3x + 2}{x - 2} = x - 1$ is not valid when $x = 2$, since it involves division by zero at this point. However, it is valid arbitrarily close to $x = 2$, allowing us to calculate the above limit.

L'Hôpital's rule

Other indeterminate limits can be evaluated using l'Hôpital's rule.

Suppose that f and g are differentiable functions, and that $g'(x) \neq 0$ on an open interval that contains a. Suppose that

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0$$

OR $$\lim_{x \to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm\infty$$

Then, $$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Example

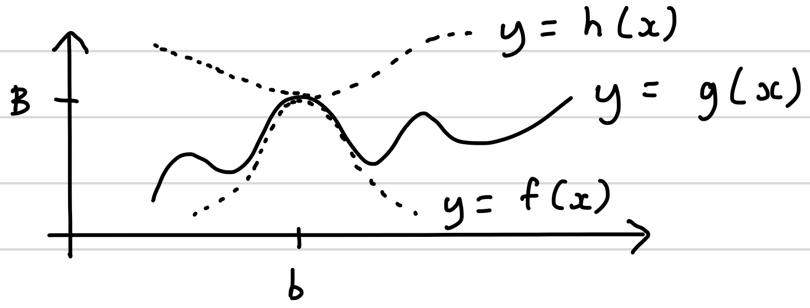
$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = \frac{1}{1} = 1$$

The sandwich theorem (or squeeze theorem)

Let $f(x) \leq g(x) \leq h(x)$ in $(a, b) \cup (b, c)$

If $\lim_{x \to b} f(x) = \lim_{x \to b} h(x) = B$, then

$$\lim_{x \to b} g(x) = B$$



<u>Example</u>

Evaluate $\lim_{x \to 0^+} x \sin(1/x)$

We know that the value of the sine function always lies between -1 and 1 inclusive, so that

$$-1 \leq \sin(1/x) \leq 1$$

Since $x > 0$, we may multiply this inequality through by x to get

$$-x \leq x \sin(1/x) \leq x$$

This is in the form $f(x) \leq g(x) \leq h(x)$.

Since $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (-x) = 0$

and $\lim_{x \to 0^+} h(x) = \lim_{x \to 0^+} (x) = 0$,

we also have that $\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} x \sin(1/x)$

$$= 0$$

by the sandwich theorem.

Limits involving rational functions as $x \to \infty$

Example If we try to calculate

$$\lim_{x \to \infty} \frac{x - 8x^4}{7x^4 + 5x^3 + 2000x^2 - 6}$$ by direct substitution, we arrive at the indeterminate form ∞ / ∞.

- To evaluate this, we divide both the numerator and denominator by the highest power of x (the leading-order term) and then take the limit.
- We have

$$\lim_{x \to \infty} \frac{x - 8x^4}{7x^4 + 5x^3 + 2000x^2 - 6}$$

$$= \lim_{x \to \infty} \frac{x/x^4 - 8x^4/x^4}{7x^4/x^4 + 5x^3/x^4 + 2000x^2/x^4 - 6/x^4}$$

$$= \lim_{x \to \infty} \frac{1/x^3 - 8}{7 + 5/x + 2000/x^2 - 6/x^4}$$

$$= \frac{0 - 8}{7 + 0 - 0 - 0} = \frac{-8}{7}$$

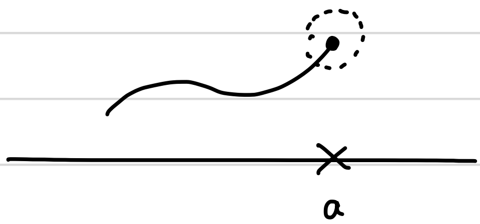
Continuity - informal definition

The graph of a continuous function can be drawn without removing your pen from the paper.

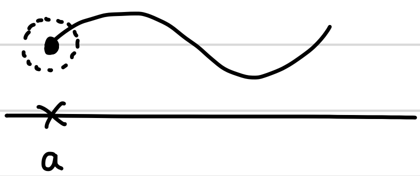
Continuity at a point - formal definition

Let f be defined on an interval that includes the point a.

f is <u>continuous from the left</u> at a if $\lim\limits_{x \to a^-} f(x) = f(a)$.



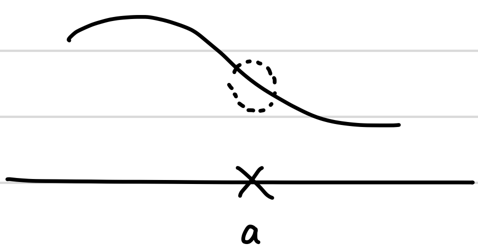
f is <u>continuous from the right</u> at a if $\lim\limits_{x \to a^+} f(x) = f(a)$



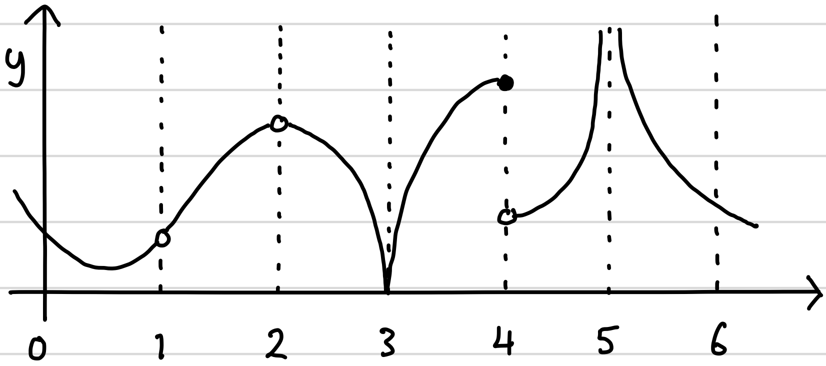
f is <u>continuous</u> at a if $\lim\limits_{x \to a} f(x) = f(a)$

i.e. if $\lim\limits_{x \to a^-} f(x) = f(a)$

and $\lim\limits_{x \to a^+} f(x) = f(a)$



Example The function with the graph



$y = f(x)$

is continuous from the left at $x = 3$, $x = 4$ and $x = 6$, and continuous from the right at $x = 3$ and $x = 6$. This means that it is continuous at $x = 3$ and $x = 6$.

Types of discontinuity

- The discontinuities at $x = 1$ and $x = 2$ are removable discontinuities: the limits of f at $x \to 1$ and $x \to 2$ exist, but are not equal to the values of f at the respective points. They are called removable because they can be removed by redefining the function at a single point.
- The discontinuity at $x = 4$ is a jump discontinuity.
- The discontinuity at $x = 5$ is an infinite discontinuity.

Testing for continuity

1. Check whether $f(x)$ is defined at $x = a$.
2. Check whether $\lim_{x \to a} f(x)$ exists.
3. Check whether $\lim_{x \to a} f(x) = f(a)$.

Examples

Determine whether the following functions are continuous at $x = 2$

(a) $$f(x) = \frac{x^2 - 4}{x - 2}$$

This function is undefined at $x = 2$, so cannot be continuous there

(b) $$g(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & \text{when } x \neq 2 \\ 4 & \text{when } x = 2 \end{cases}$$

Now, $g(x)$ is defined at $x = 2$. We further see that

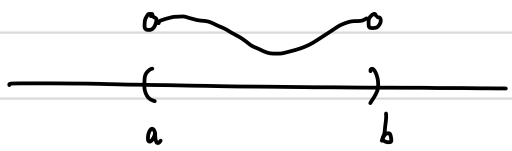
$$\lim_{x \to 2} g(x) = \lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x-2)(x+2)}{x-2}$$

$$= \lim_{x \to 2} x + 2 = 4.$$

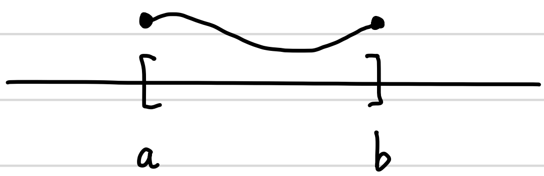
$\Rightarrow$ $g(x)$ is continuous at $x = 2$. This means that the discontinuity in $f(x)$ was removable, and was removed in $g(x)$ by defining the function separately at $x = 2$.

Continuity on intervals

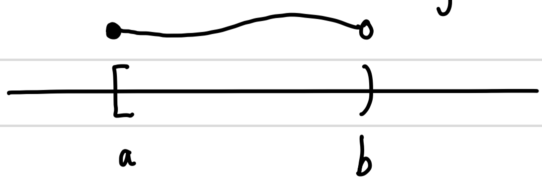
- A function f is continuous on an open interval (a, b) if it is continuous at every point in the interval.



- A function f is continuous on a closed interval $[a, b]$ if it is
  - continuous on (a, b)
  - continuous from the right at a
  - continuous from the left at b.



- A function f is continuous on a half-open interval $[a, b)$ if it is
  - continuous on (a, b), and
  - continuous from the right at a.



Continuity of combinations of functions

Suppose that the functions $f(x)$ and $g(x)$ are continuous on $[a, b]$ and that c is a real constant. Then,

1. $cf(x)$ is continuous on $[a, b]$
2. $f(x) \pm g(x)$ is continuous on $[a, b]$
3. fg is continuous on $[a, b]$.
4. $\frac{f}{g}$ is continuous on $[a, b]$ provided that $g(x) \neq 0 \quad \forall x \in [a, b]$
5. Suppose that g is continuous at c and that f is continuous at $g(c)$. Then, the composition $f(g(x))$ is continuous at c.

<u>Proof</u>: Let $y = g(x)$. Since $g(x)$ is continuous at c, $y \to g(c)$ as $x \to c$.

Then, $\lim_{x \to c} f(g(x)) = \lim_{y \to g(c)} f(y)$.

Since f is continuous at $g(c)$, $\lim_{y \to g(c)} f(y) = f(g(c))$.

We then have $\lim_{x \to c} f(g(x)) = f(g(c))$

and the composition is continuous at c.

The following types of functions are continuous at every number in their domains:

- Polynomials
- Rational functions
- Root functions
- Trigonometric functions

Examples

1. $g(x) = \dfrac{x^3 - 2x + 1}{x - 7}$ is a rational function

   $\Rightarrow$ it is continuous on its domain $\{x \in \mathbb{R} : x \neq 7\}$.

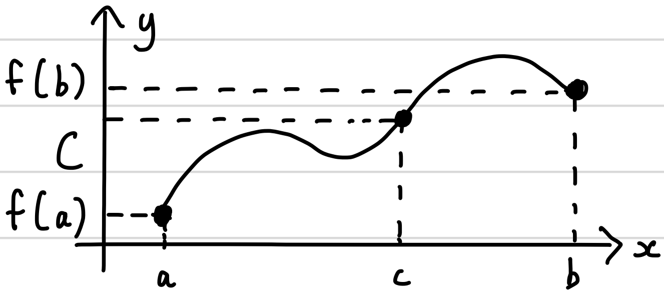
2. $h(x) = \sqrt{x} + \dfrac{1}{x - 1}$

We can write $h(x) = G(x) + H(x)$, with $G(x) = \sqrt{x}$ and $H(x) = \dfrac{1}{x-1}$. Then $G(x)$ is continuous on its domain, $[0, \infty)$.

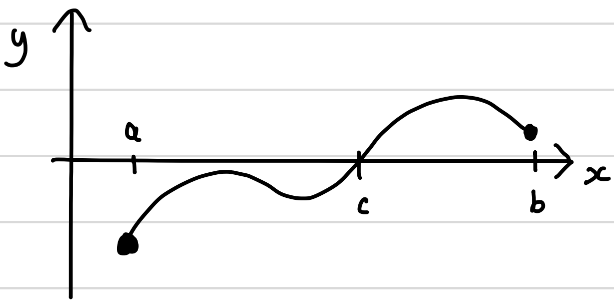
$H(x)$ is also continuous on its domain; i.e., everywhere, except at $x = 1$. So, $h(x)$ is continuous on the intervals $[0, 1)$ and $(1, \infty)$.

Intermediate value theorem

If f is a real-valued continuous function on $[a, b]$, then, for every $f(a) \leqslant C \leqslant f(b)$, or $f(b) \leqslant C \leqslant f(a)$, there exists at least one $c \in [a, b]$ such that $f(c) = C$.



Special case: $C = 0$. This can be used for root finding:



If f is continuous on $[a, b]$, and $f(a)$ and $f(b)$ have opposite signs, then the equation $f(x) = 0$ has at least one solution on (a, b).

Example

Show that the equation $17x^7 - 19x^5 - 1$ has a solution between -1 and 0.

Let $f(x) = 17x^7 - 19x^5 - 1.$

Now $f(-1) = 17(-1)^7 - 19(-1)^5 - 1$

$= -17 + 19 - 1$

$= 1$

and $f(0) = 17(0)^7 - 19(0)^5 - 1$

$= -1$

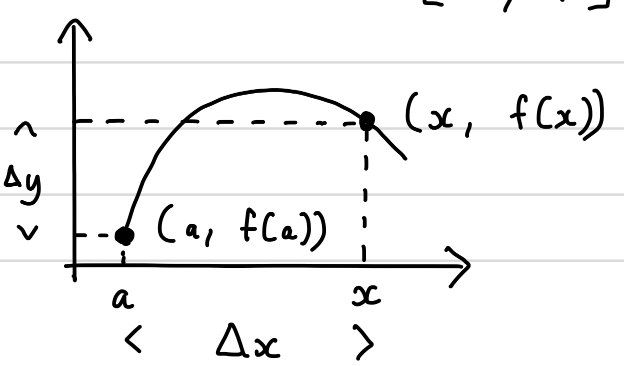
Therefore, since $f(-1)$ and $f(0)$ have opposite signs and $f(x)$ is a polynomial (a continuous function), $f(x) = 0$ has a solution between -1 and 0.

(This is called bracketing the root, and is an important first step when solving equations on a computer.)

<u>Differentiation</u>

- A basic quantity in differential calculus is the <u>difference quotient</u>

If $y = f(x)$ is a continuous function on the interval $[a, x]$, then the average rate of change of y with respect to x on the interval $[a, x]$ is $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x) - f(a)}{x - a}$



- The average rate of change does not tell us anything about the details of what happened between the two endpoints.
- In contrast, the <u>derivative</u> tells us the rate of change at a point.

Differentiability of functions at a point

Definition: differentiability

Suppose that $f(x)$ is defined on an interval (a, b) containing the point x_0 (i.e. $a < x_0 < b$). Then, $f(x)$ is differentiable at x_0 if and only if the following limit exists:

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

- If this limit exists, it defines the derivative of f with respect to x at the point x_0, written $f'(x_0)$.
- The derivative at a point gives the gradient of the tangent at that point.
- It gives the instantaneous rate of change of $f(x)$ with respect to x at $x = x_0$.

Derivative at a general point (from first principles)

Definition: If $f(x)$ is differentiable at every point in its domain, then it is a differentiable function. From first principles, the derivative of the function with respect to x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- If $y = f(x)$, we write $f'(x) = dy/dx$.

Examples

- If $f(x) = x^n$, $f'(x) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$

$$= \lim_{h \to 0} \frac{x^n + nx^{n-1}h + {}^nC_2\, x^{n-2}h^2 + \dots + h^n - x^n}{h}$$

$$= \lim_{h \to 0} \frac{nx^{n-1}h + {}^nC_2\, x^{n-2}h^2 + \dots + h^n}{h}$$

$$= \lim_{h \to 0} nx^{n-1} + {}^nC_2\, x^{n-2}h + \dots + h^{n-1}$$

$$= nx^{n-1}$$

- This result can be used to derive other standard results:

$$\frac{d}{dx} e^x = \frac{d}{dx}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right)$$

$$= 0 + 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$= e^x$$

$$\frac{d}{dx}(\sin x) = \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \cos x$$

$$\frac{d}{dx}(\cos x) = \frac{d}{dx}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)$$

$$= 0 - x + \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots = -\sin x$$

- If $f(x) = 1/x$, $f'(x) = \lim_{h \to 0} \frac{1}{h}\left(\frac{1}{x+h} - \frac{1}{x}\right)$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{x-(x+h)}{x(x+h)}\right) = \lim_{h \to 0} \frac{1}{h}\left(-\frac{h}{x(x+h)}\right)$$

$$= \lim_{h \to 0} -\frac{1}{x(x+h)} = -\frac{1}{x^2}$$

- If $f(x) = \sqrt{x}$,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$