Noting that $e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$
$= \cos\theta - i\sin\theta$,
we also find that $e^{i\theta} + e^{-i\theta} = 2\cos\theta$

and $$\boxed{\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}}$$

Similarly, $e^{i\theta} - e^{-i\theta} = 2i\sin\theta$

and $$\boxed{\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}}$$

Note: this makes clear the connection between the trigonometric functions and the hyperbolic functions $\cosh x = (e^{x} + e^{-x})/2$ and $\sinh x = (e^{x} - e^{-x})/2$.

De Moivre's theorem

From $z = re^{i\theta}$ with $e^{i\theta} = \cos\theta + i\sin\theta$, it follows that
$$z^n = r^n e^{in\theta} = r^n\{\cos(n\theta) + i\sin(n\theta)\}$$

so that $$\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta}$$

Example: Use de Moivre's theorem to derive the double angle formulas.

Put $n = 2$. Then,

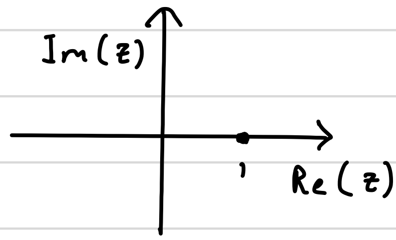
$$(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$$
$$\cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta = \cos 2\theta + i\sin 2\theta$$

Equate the real and imaginary parts:

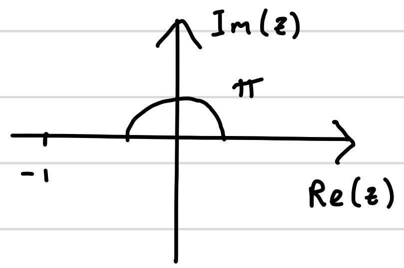
$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\sin 2\theta = 2\sin\theta\cos\theta$$

More examples of polar and exponential form

a) $1 = 1 + 0i$
$= 1(\cos 0 + i\sin 0)$
$= e^{i0}$



b) $-1 = 1[\cos(\pi) + i\sin(\pi)]$
$= e^{i\pi}$

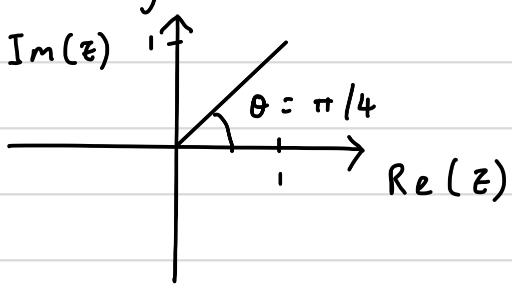


c) $z = 1 + i$

Modulus : $|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$

Argument: $\theta = \tan^{-1}(1/1) = \tan^{-1}(1)$

This could be $\pi/4$ or $5\pi/4$. However, we see from the Argand diagram that it must be $\pi/4$:



Then, $z = \sqrt{2}\,[\cos(\pi/4) + i\sin(\pi/4)]$

$$= \sqrt{2}\, e^{i\pi/4}$$

We can also write this as

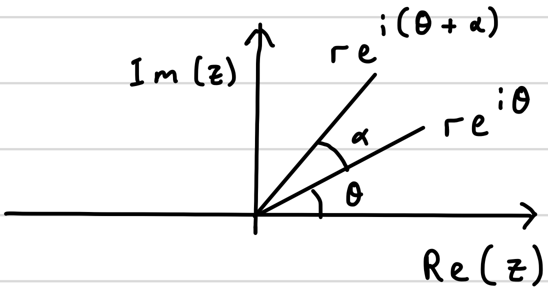
$$z = \sqrt{2}\, e^{i(\pi/4 + 2k\pi)} \quad \text{with } k \in \mathbb{Z}$$

since adding 2π to the argument brings us back to the same point in the Argand diagram.

Rotation of a complex number

If $z = re^{i\theta}$, then $ze^{i\alpha} = re^{i\alpha}e^{i\theta} = re^{i(\alpha+\theta)}$

i.e. multiplying a complex number by $e^{i\alpha}$ rotates it by an angle α in the complex plane.



Multiplication in exponential form

Multiplication and division are neater in exponential form than in standard form. We have that, for $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$,

$$z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

We can see that $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$. Since $|e^{i\theta}| = 1$ for all θ, we also have that $|z_1 z_2| = r_1 r_2 = |z_1||z_2|$.

Division is similar:

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$ and we see that

$|z_1/z_2| = |z_1|/|z_2|$ and $\arg(z_1/z_2) = \arg z_1 - \arg z_2$

Roots of complex numbers

- Finding the n^{th} root of a complex number w corresponds to solving

$$z^n = w$$

- Key fact: adding an integer multiple of 2π to the argument of a complex number leaves the complex number unchanged.

Step 1 : Write w in exponential form:
$w = |w| e^{i\phi}$

Step 2 : Add $2k\pi$, $k \in \mathbb{Z}$, to the argument of w. This leaves w unchanged.

$$w = |w| e^{i\phi + 2k\pi i}$$

Step 3 : Write $z^n = w$, or

$$z^n = |w| e^{i\phi + 2k\pi i}$$

Step 4 : Take the n^{th} root of both sides:

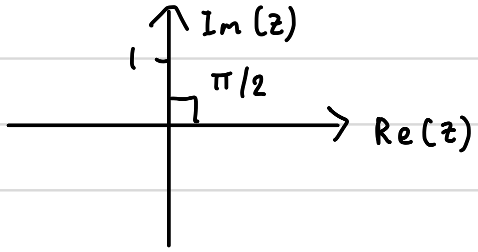
$$z = |w|^{1/n} e^{i\phi/n + 2k\pi i/n}$$

Step 5 : Let $k = 0, 1, 2, 3, \dots, n-1$ to read off the n roots.
Note: we stop at $n-1$ as $k = n$ gives the same root as $k = 0$.

Example

$$z^3 = i$$

1 Write i in exponential form:



$$|i| = 1$$
$$\arg(i) = \pi/2$$
and $i = e^{i\pi/2}$

2 Add $2k\pi$, $k \in \mathbb{Z}$, to the argument of i:

$$e^{i\pi/2 + 2k\pi i}$$

3 Write $z^3 = i = e^{i\pi/2 + 2k\pi i}$

(4) Take the 3rd (cube) root of both sides

$$z = e^{i\pi/6 + 2k\pi i/3}$$

(5) Let $k = 0, 1, 2$:

$k = 0$: $z = e^{\pi i/6}$
$k = 1$: $z = e^{i\pi/6 + 2\pi i/3} = e^{i\pi/6 + 4\pi i/6} = e^{5\pi i/6}$
$k = 2$: $z = e^{i\pi/6 + 4\pi i/3} = e^{i\pi/6 + 8\pi i/6} = e^{9\pi i/6} = e^{3\pi i/2}$

These are the three roots. When $k = 3$, $z = e^{i\pi/6 + 6\pi i/3} = e^{i\pi/6 + 2\pi i} = e^{i\pi/6}$, the first root again.

<u>Limits</u>

- For many functions $f(x)$, the value of $f(x)$ as x approaches the value a will simply be $f(a)$.
- However, the function could be undefined at the point a.

<u>Example</u>: the function

$$f(x) = \frac{\sin x}{x},$$

used in optics and signal processing, is undefined at $x = 0$.
- In cases like this, we need the concept of a <u>limit</u>.

Notation and definition

Suppose $f(x)$ is defined when x is near to the number a, except possibly at a itself. If we can make $f(x)$ arbitrarily close to L by making x sufficiently close to a (on either side of it but not equal to it), then we write

$$\lim_{x \to a} f(x) = L$$

and say that the limit of $f(x)$ as x approaches a is equal to L.

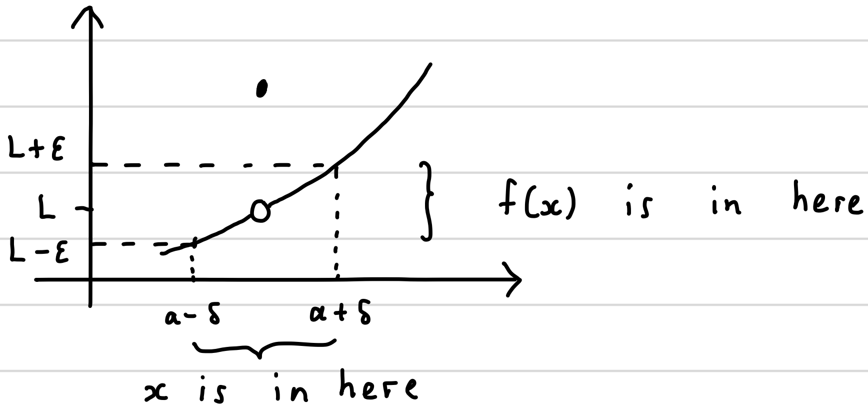
Precise definition

Suppose that $f(x)$ is defined on an open interval that contains the number a, except possibly at a itself. Then, we say that the limit of $f(x)$ as x approaches a is L, and we write

$$\lim_{x \to a} f(x) = L$$

if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that, if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

Note: this definition is needed in rigorous proofs: it uses variables, ε and δ, rather than statements like "arbitrarily close".



One-sided limits

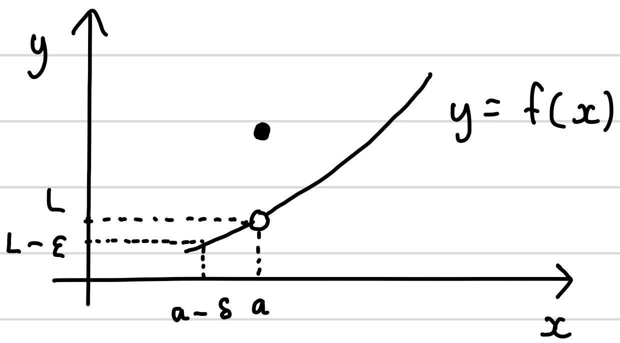
Left-sided limit

We write $\lim_{x \to a^-} f(x) = L$

and say that the limit of $f(x)$ as x approaches a from the left is equal to L if we can make the value of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and x is less than a.

OR $\lim_{x \to a^-} f(x) = L$

if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that, if $a - \delta < x < a$, then $|f(x) - L| < \varepsilon$.



Right-sided limit

$$\lim_{x \to a^+} f(x) = L$$

if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that, if $a < x < a + \delta$, then $|f(x) - L| < \varepsilon$.

Note that $\lim_{x \to a} f(x) = L$ if and only if

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L.$$

Simple examples of limits

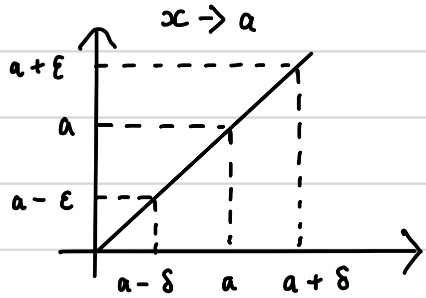
① $f(x) = A$, where A is a constant

Then, $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = \lim_{x \to a} f(x) = A$

② $f(x) = x$

$$\lim_{x \to a^-} f(x) = a, \quad \lim_{x \to a^+} f(x) = a$$

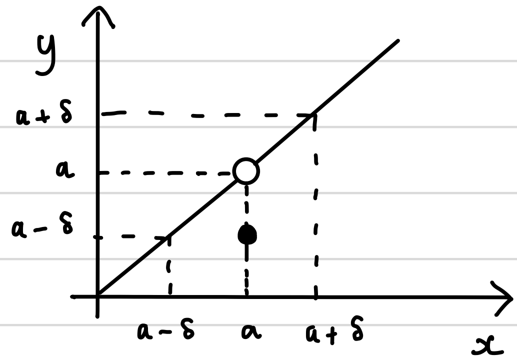
and $\lim_{x \to a} f(x) = a$



In this case, $\delta = \varepsilon$.

$\lim_{x \to a} f(x)$ versus $f(a)$

Consider the function $f(x) = \begin{cases} x & x < a \\ a/2 & x = a \\ x & x > a \end{cases}$



$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

$$= \lim_{x \to a} f(x) = a \neq f(a)$$

- The limit of $f(x)$ at a does not depend on the existence of $f(x)$ at a or, when $f(a)$ exists, on the value of $f(a)$.

Step functions and limits

- The signum function discussed earlier is sometimes written as $f(x) = \dfrac{x}{|x|}$.

Does this function have a limit as $x \to 0$? We see that $\lim_{x \to 0^-} f(x) = -1$ and that

$\lim_{x \to 0^+} f(x) = 1$. Since $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$,

the limit $\lim_{x \to 0} f(x)$ does not exist.

However, the signum function is sometimes defined to have $f(0) = 0$.

Limits at infinity.

The limits at infinity of $f(x)$ describe its behaviour as x increases or decreases without bound.
Example If $f(x) = 1/x$, then $\lim_{x \to \infty} f(x) = 0$ and

$$\lim_{x \to -\infty} f(x) = 0.$$

Infinite limits

Some functions become infinite as x tends to a finite value

If $f(x) = \dfrac{1}{x}$, then $\lim_{x \to 0^-} f(x) = -\infty$

and $\lim_{x \to 0^+} f(x) = \infty$

Rules for limits

Let $\lim_{x \to a} f(x) = F$ and $\lim_{x \to a} g(x) = G$

Then, $\lim_{x \to a} (k f(x)) = kF$

$$\lim_{x \to a} (f(x) \pm g(x)) = F \pm G$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{F}{G} \qquad (G \neq 0)$$

$$\lim_{x \to a} f(x)\, g(x) = FG$$

If $f(x) \leq g(x)$ on an interval containing a, then $\lim\limits_{x \to a} f(x) \leq \lim\limits_{x \to a} g(x)$.

If $P(x)$ and $Q(x)$ are polynomials, then

$$\lim_{x \to a} P(x) = P(a), \quad \lim_{x \to a} Q(x) = Q(a)$$

and $$\lim_{x \to a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)} \qquad Q(a) \neq 0$$