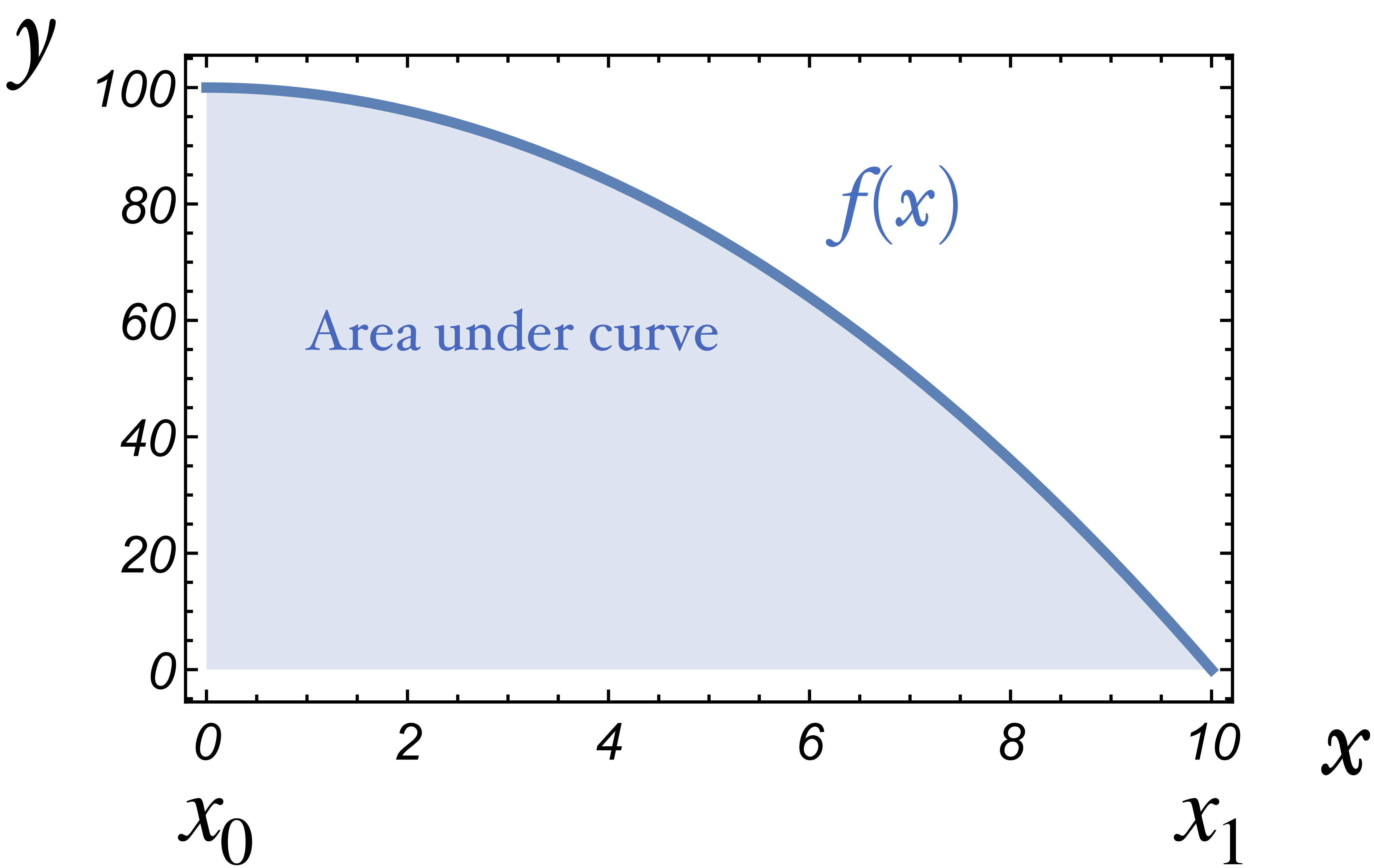
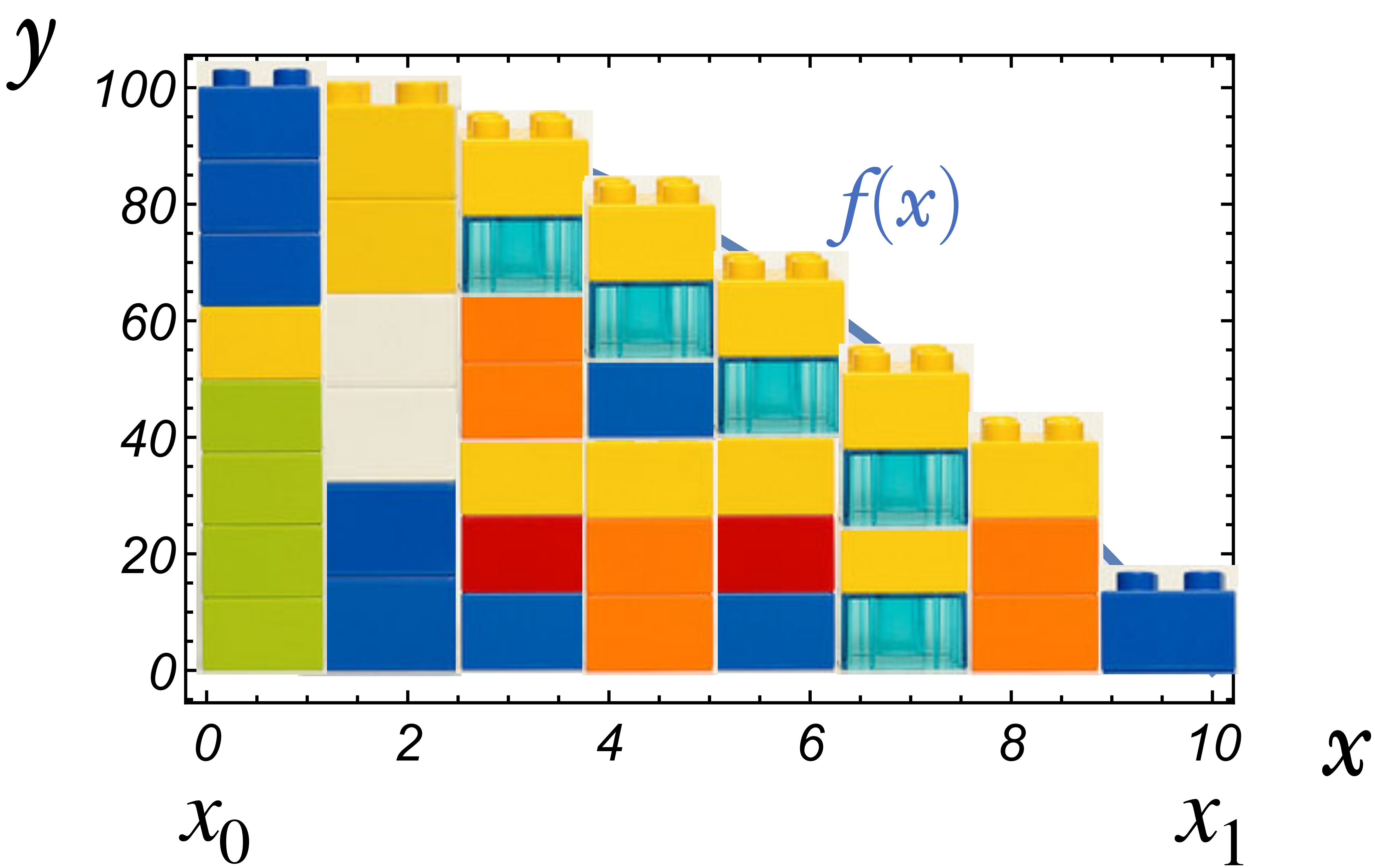
Additional notions of calculus

Riemann integral

Look at the area under a curve

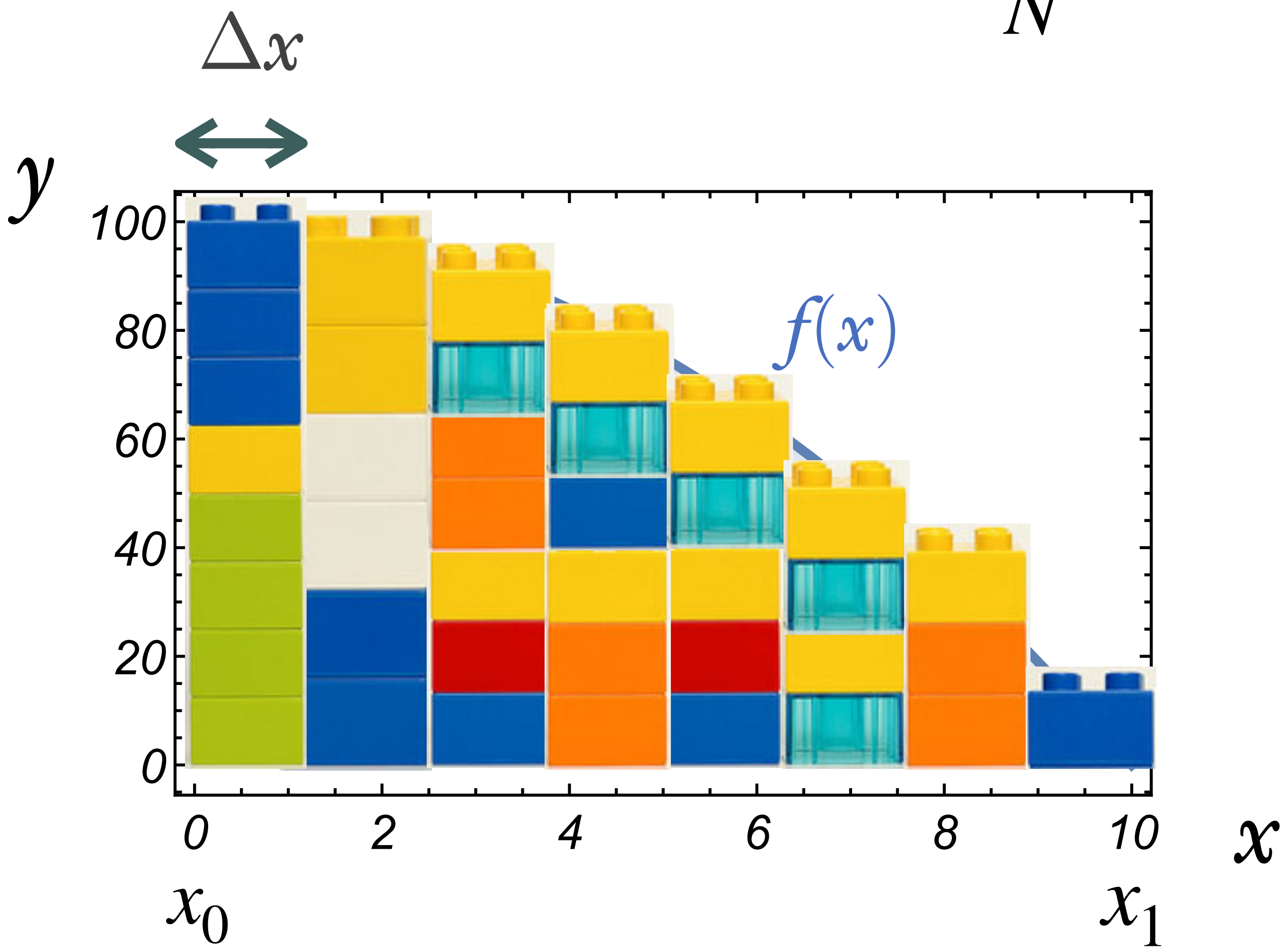


Look at the area under a curve



Look at the area under a curve

$$\Delta x = \frac{x_1 - x_0}{N}, \text{ where } N \text{ is the number of rectangles}$$



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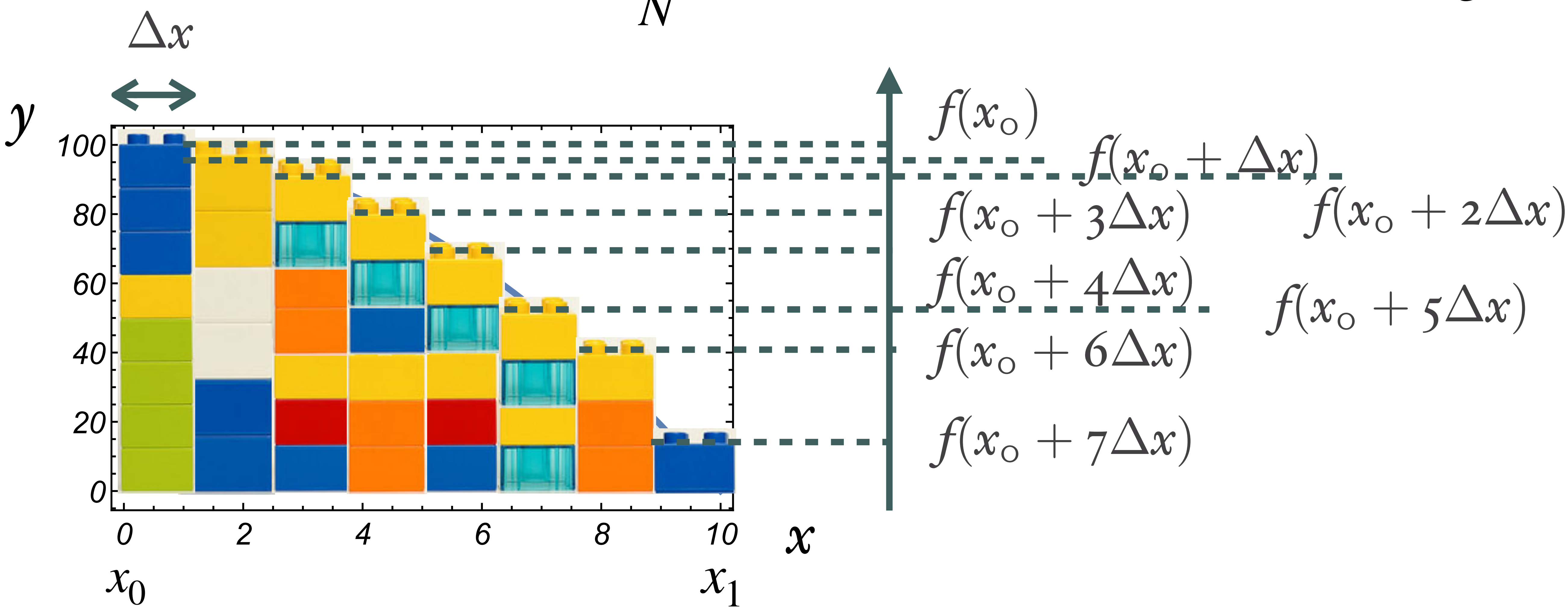


Area of the $k+1$ rectangle (height x width): $f(x_0 + k\Delta x)\Delta x$.

We set $A_N[f](x_0, x_1) = \sum_{k=0}^{N-1} f(x_0 + k\Delta x)\Delta x$ *(Riemann sum)*

Riemann integral and the fundamental theorem of calculus

Let f be a continuous function on $[x_0, x_1]$ and F a primitive function of f in $[x_0, x_1]$. Then

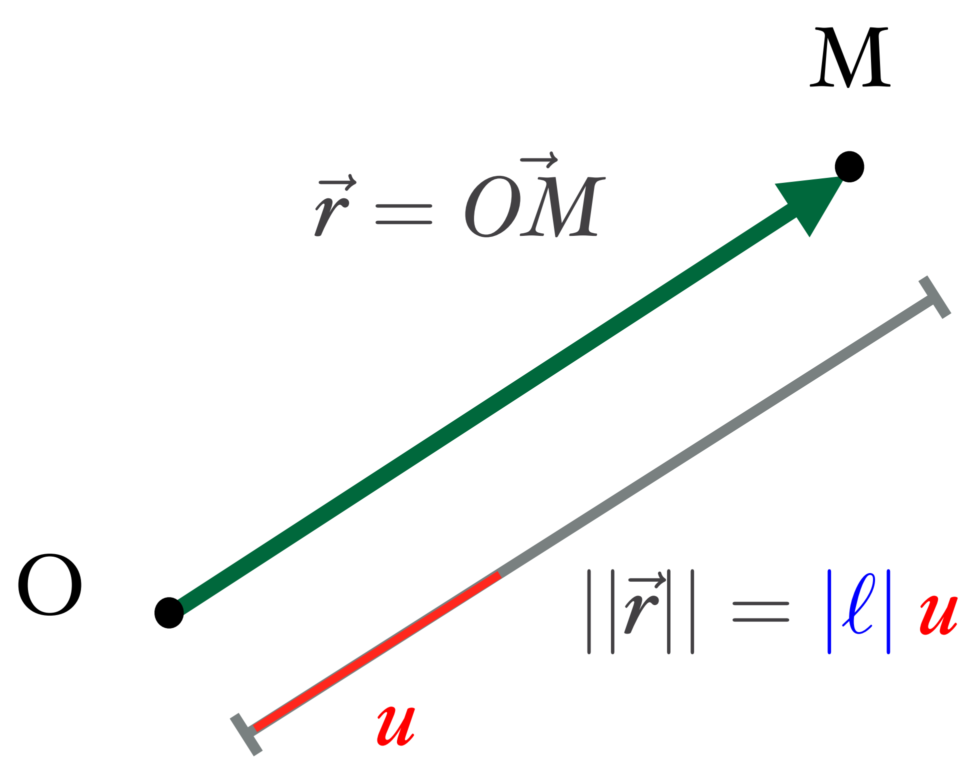
☆ $$\int_{x_0}^{x_1} f(x)dx = \lim_{N\to\infty} A_N[f](x_0, x_1)$$ *(Riemann or definite integral)*

$$\boxed{\int_{x_0}^{x_1} f(x)dx = F(x_1) - F(x_0)}$$

Introduction to kinematics in 2D

Representation of a point in 2D

In 2D the position of a point M relative to a reference point O is identified by a position vector $\vec{r} = \overrightarrow{OM}$.

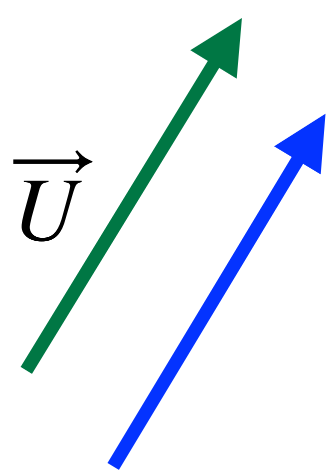


The position vector has a ***magnitude*** or ***norm*** $||\vec{r}||$ associated to a positive number $|\ell|$ related to some length unit u = cm, inches, feet, etc...

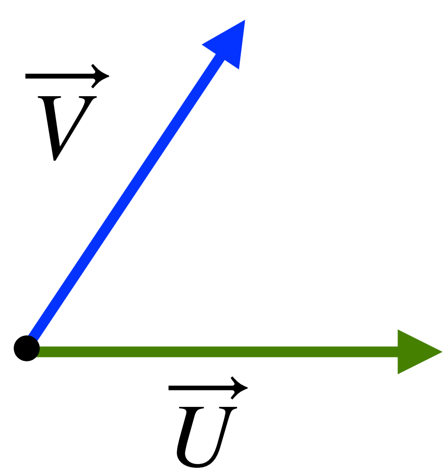
Contrary to 1D not all position vectors are proportional to each others!

A short reminder on vector algebra

☆ $\vec{U} = \vec{V}$ if and only if $\|\vec{U}\| = \|\vec{V}\|$ and $\vec{U}, \vec{V}$ point in the same direction along parallel lines.



$$\Rightarrow \quad \vec{U} = \vec{V}$$

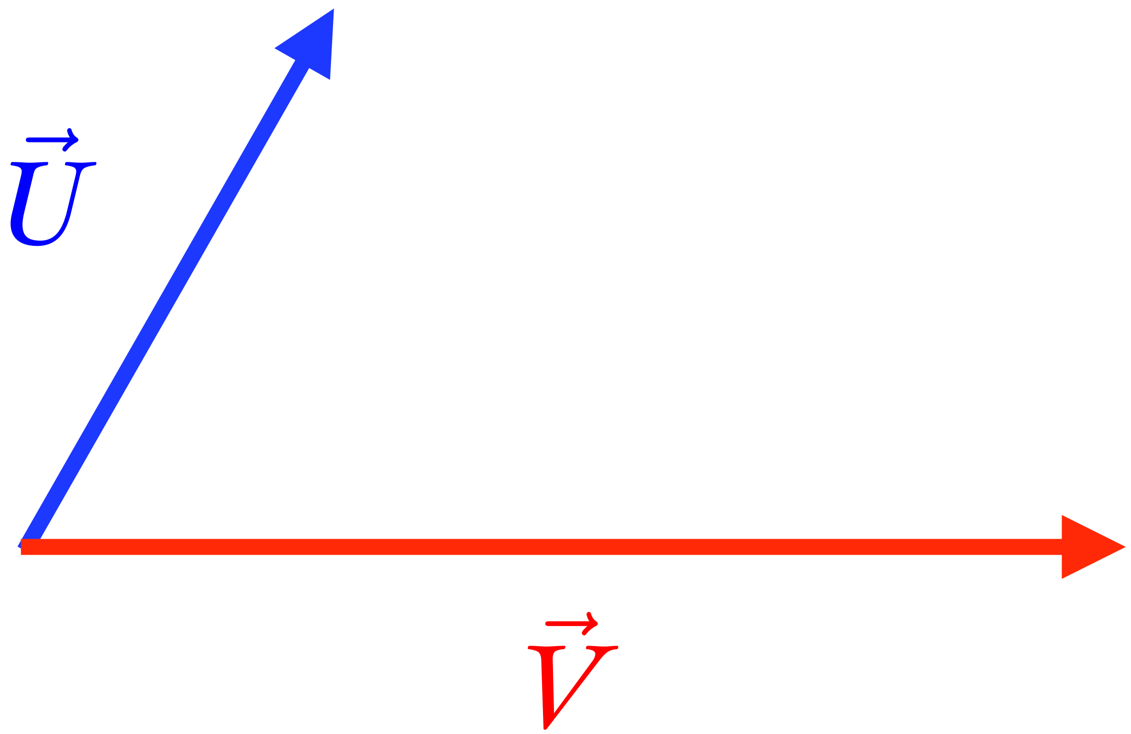


$$\Rightarrow \quad \vec{U} \neq \vec{V}$$

A short reminder on vector algebra

Addition

$\star\ \vec{U} + \vec{V}$



A short reminder on vector algebra

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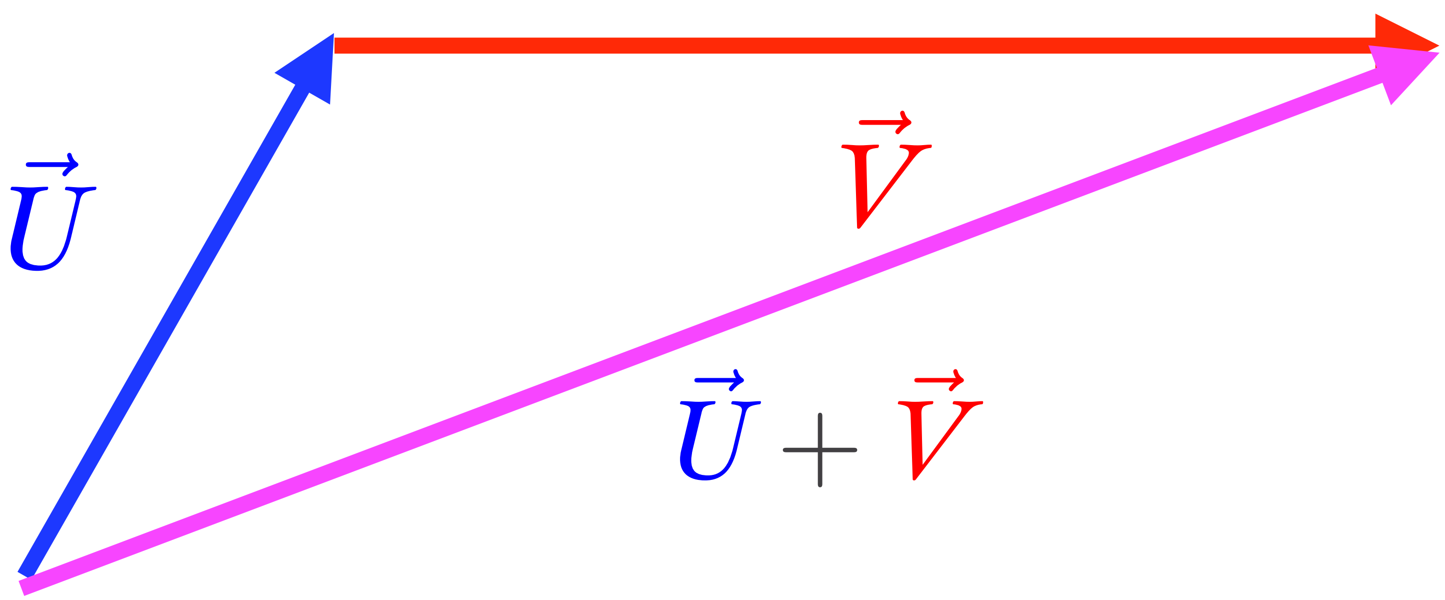
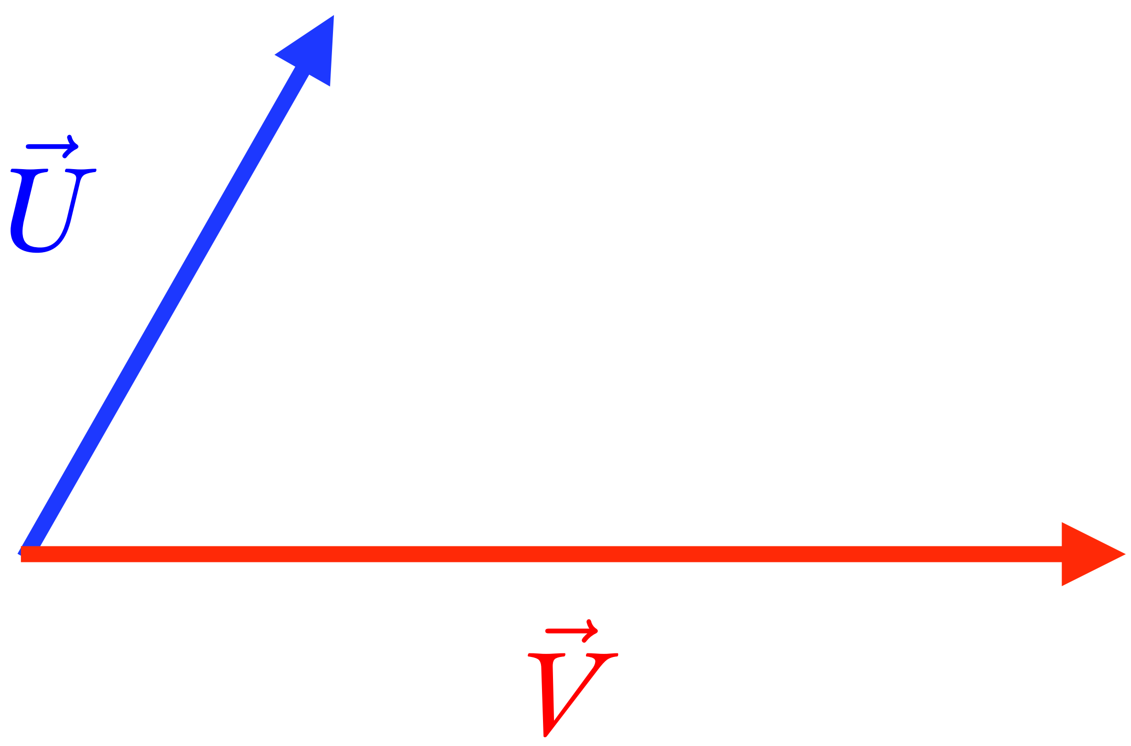


A short reminder on vector algebra

Addition

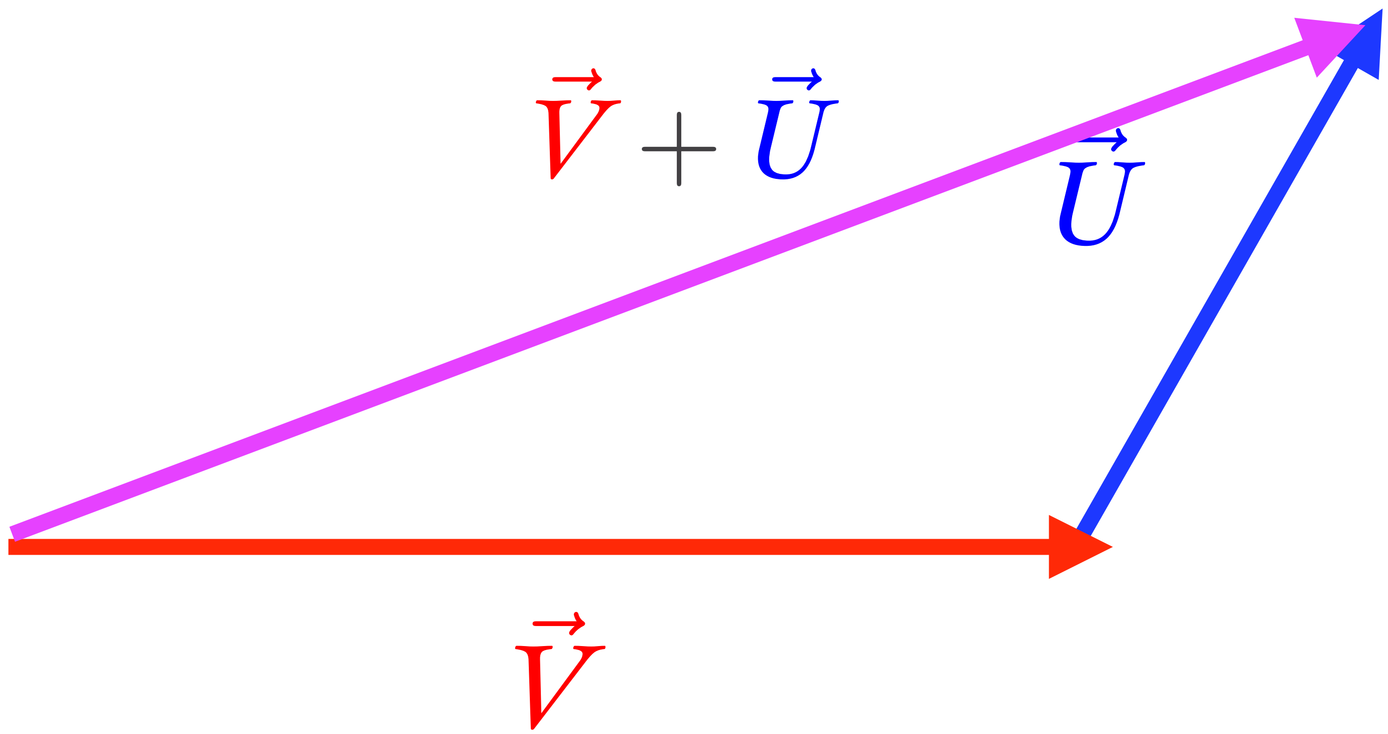


$\star\ \vec{U} + \vec{V}$

$\star\ \vec{V} + \vec{U}$

A short reminder on vector algebra

Addition

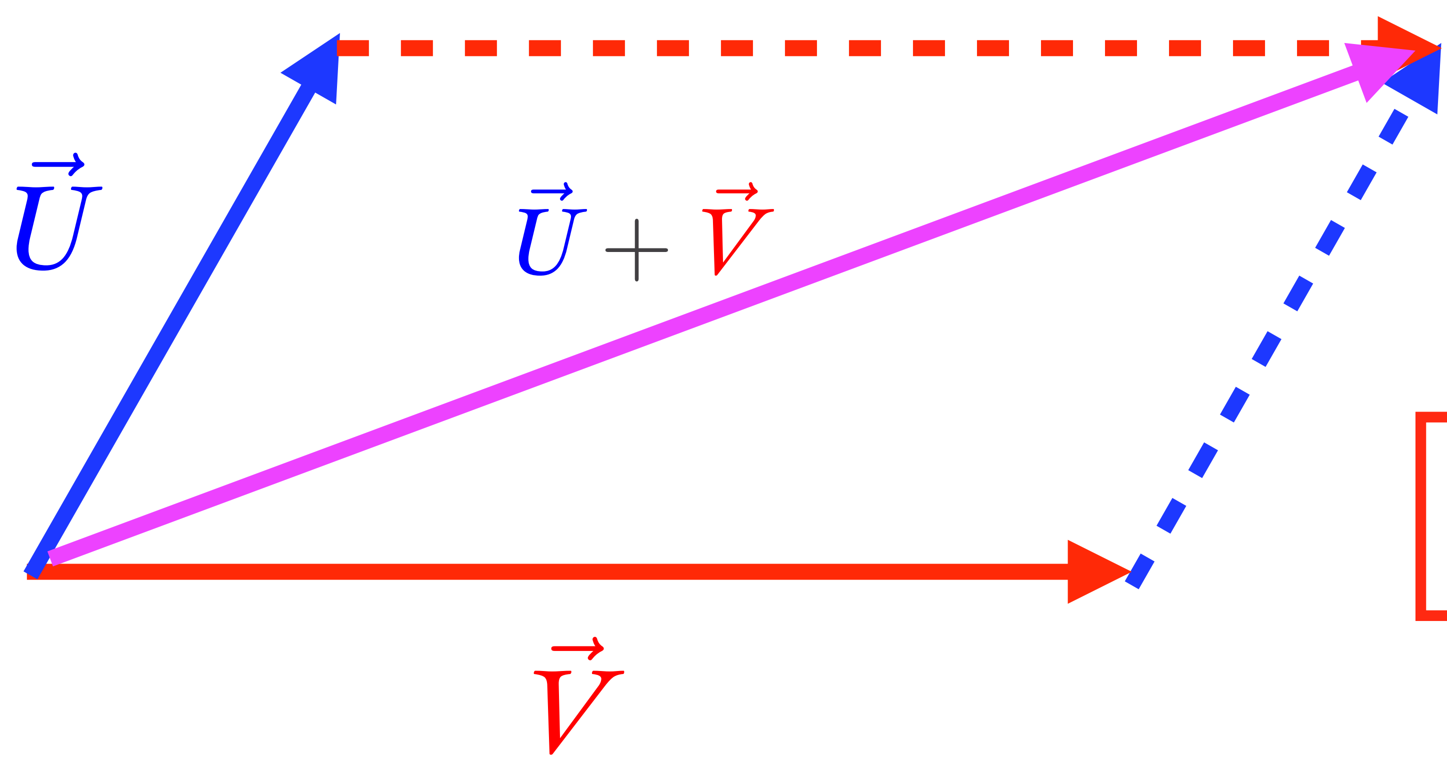


- $\star\ \vec{U} + \vec{V}$
- $\star\ \vec{V} + \vec{U}$

A short reminder on vector algebra

Addition

Parallelogram rule



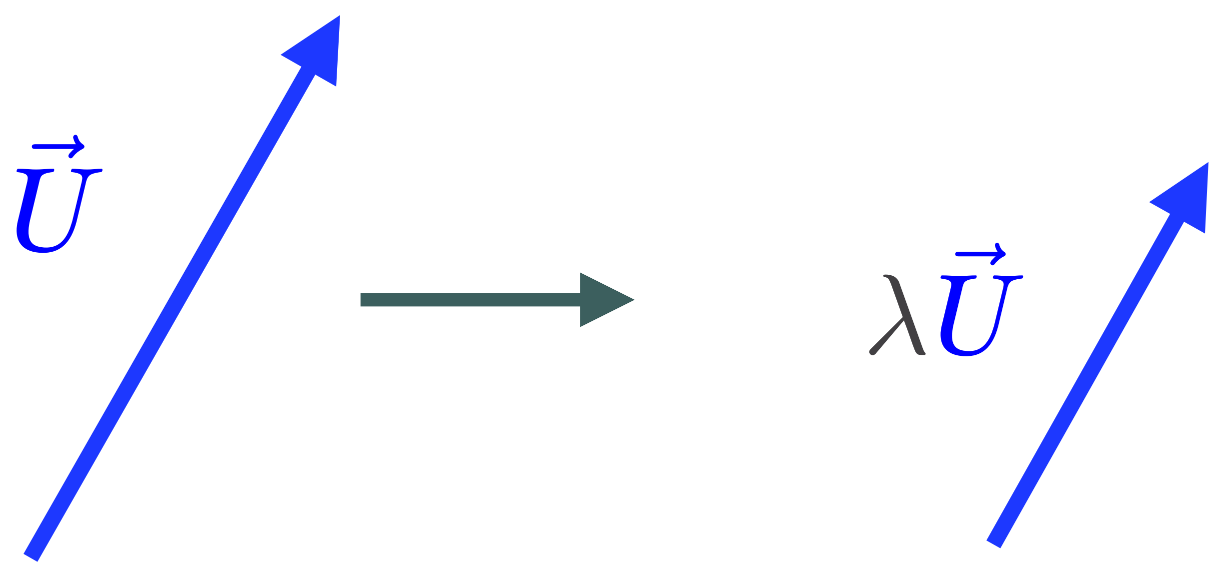
$\star\ \vec{U}+\vec{V}$

$\star\ \vec{V}+\vec{U}$

$$\boxed{\vec{U}+\vec{V}=\vec{V}+\vec{U}}$$

A short reminder on vector algebra

Multiplication by a real number



* Keep the orientation
* $||\lambda\vec{U}|| \equiv |\lambda| \times ||\vec{U}||$
* Same direction if

$$\lambda > 0$$

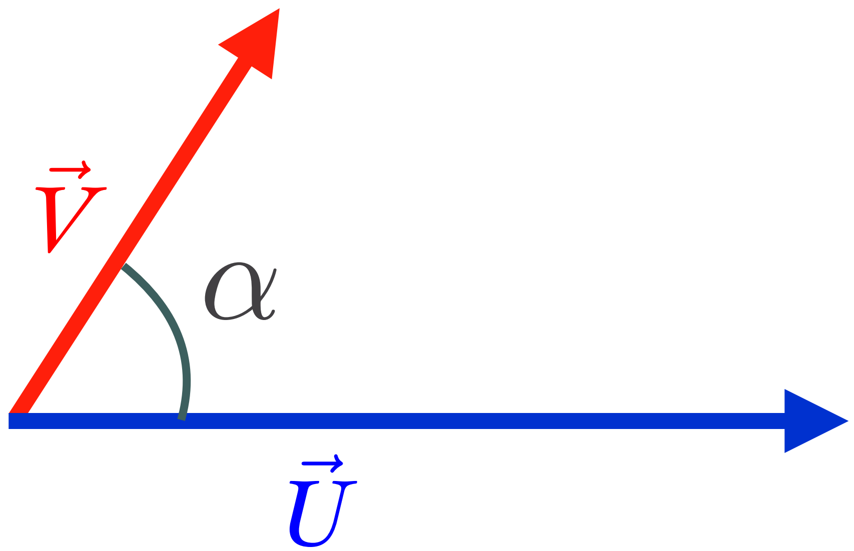
* Opposite direction if

$$\lambda < 0$$

A short reminder on vector algebra

Scalar product

$$\vec{U} \cdot \vec{V} = ||\vec{U}|| \times ||\vec{V}|| \cos\alpha \quad (\textit{scalar quantity})$$



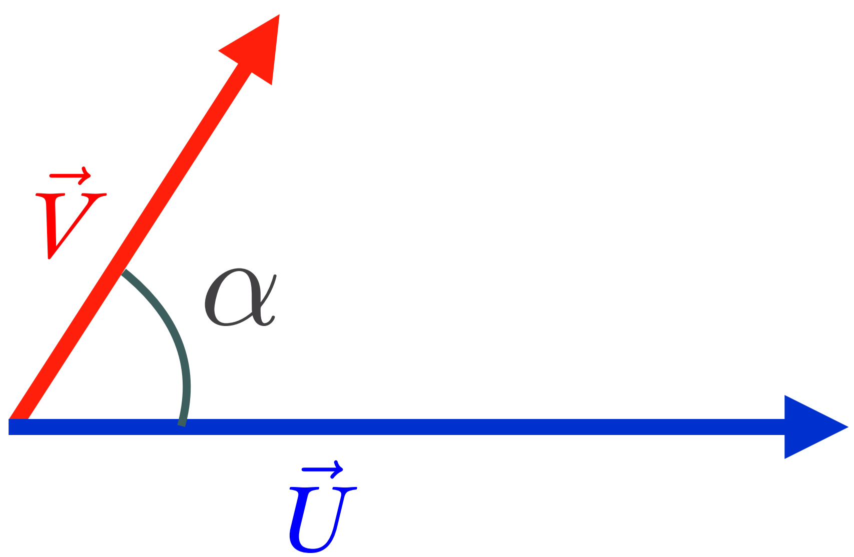
$$\boxed{\vec{V} \cdot \vec{U} = \vec{U} \cdot \vec{V}}$$

$$\boxed{||\vec{U}|| = \sqrt{\vec{U} \cdot \vec{U}}}$$

A short reminder on vector algebra

Scalar product

$$\vec{U}\cdot\vec{V} = ||\vec{U}|| \times ||\vec{V}|| \cos\alpha \quad \textit{(scalar quantity)}$$



$$\boxed{\vec{V}\cdot\vec{U} = \vec{U}\cdot\vec{V}}$$

$$\boxed{||\vec{U}|| = \sqrt{\vec{U}\cdot\vec{U}}}$$

- Two vectors $\overrightarrow{U}$ and $\overrightarrow{V}$ are said to be ***orthogonal*** or perpendicular if $\overrightarrow{U}\cdot\overrightarrow{V} = 0$, i.e. $\alpha = \frac{\pi}{2} + n\pi,\ n \in \mathbb{Z}$.
- Two vectors $\overrightarrow{U}$ and $\overrightarrow{V}$ are said to be ***collinear*** or parallel if $\overrightarrow{U}\cdot\overrightarrow{V} = \pm||\overrightarrow{U}||\,||\overrightarrow{V}||$, i.e. $\alpha = n\pi,\ n \in \mathbb{Z}$.

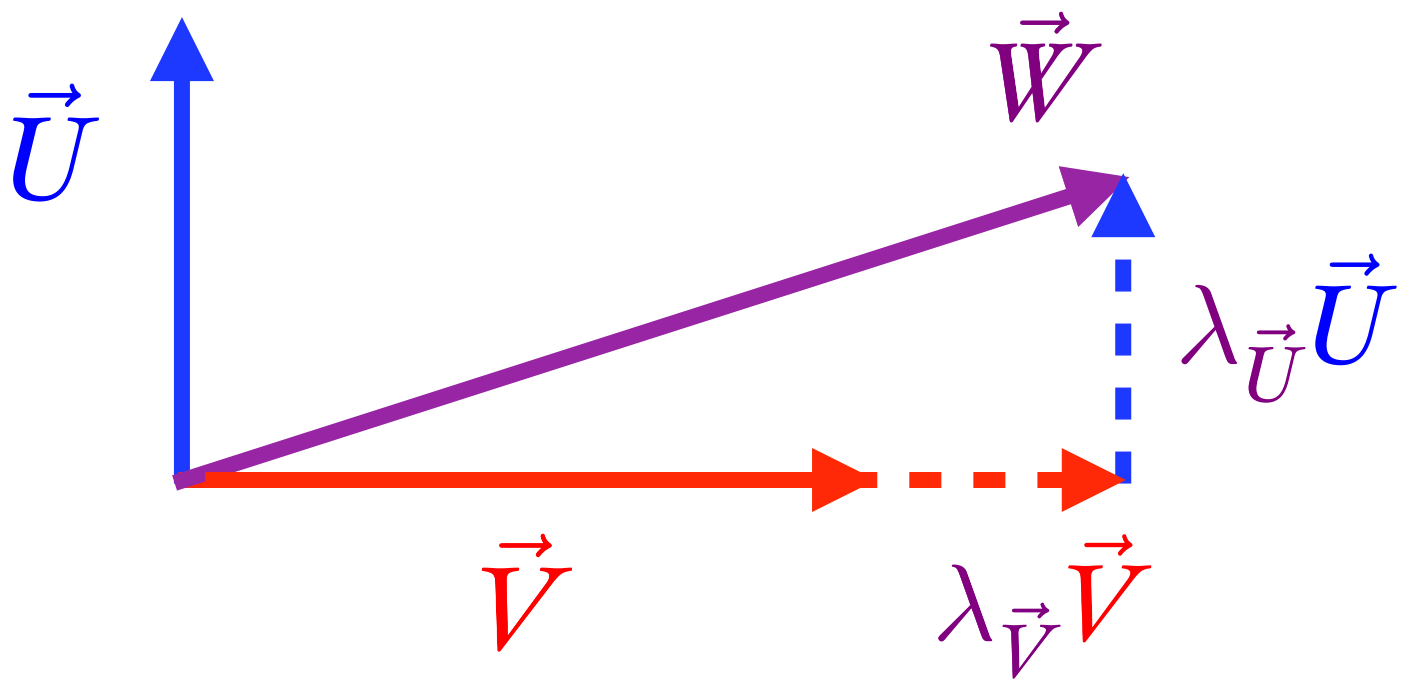
A short reminder on vector algebra

Basis and components

By the parallelogram rule, any vector in the plane can be written as the sum of two non-parallel vectors. It is in particular true with perpendicular vectors:

$$\vec{W} = \lambda_{\vec{U}}\vec{U} + \lambda_{\vec{V}}\vec{V}$$

$$\vec{W} = \lambda_{\vec{U}}||\vec{U}||\frac{\vec{U}}{||\vec{U}||} + \lambda_{\vec{V}}||\vec{V}||\frac{\vec{V}}{||\vec{V}||}$$



A short reminder on vector algebra

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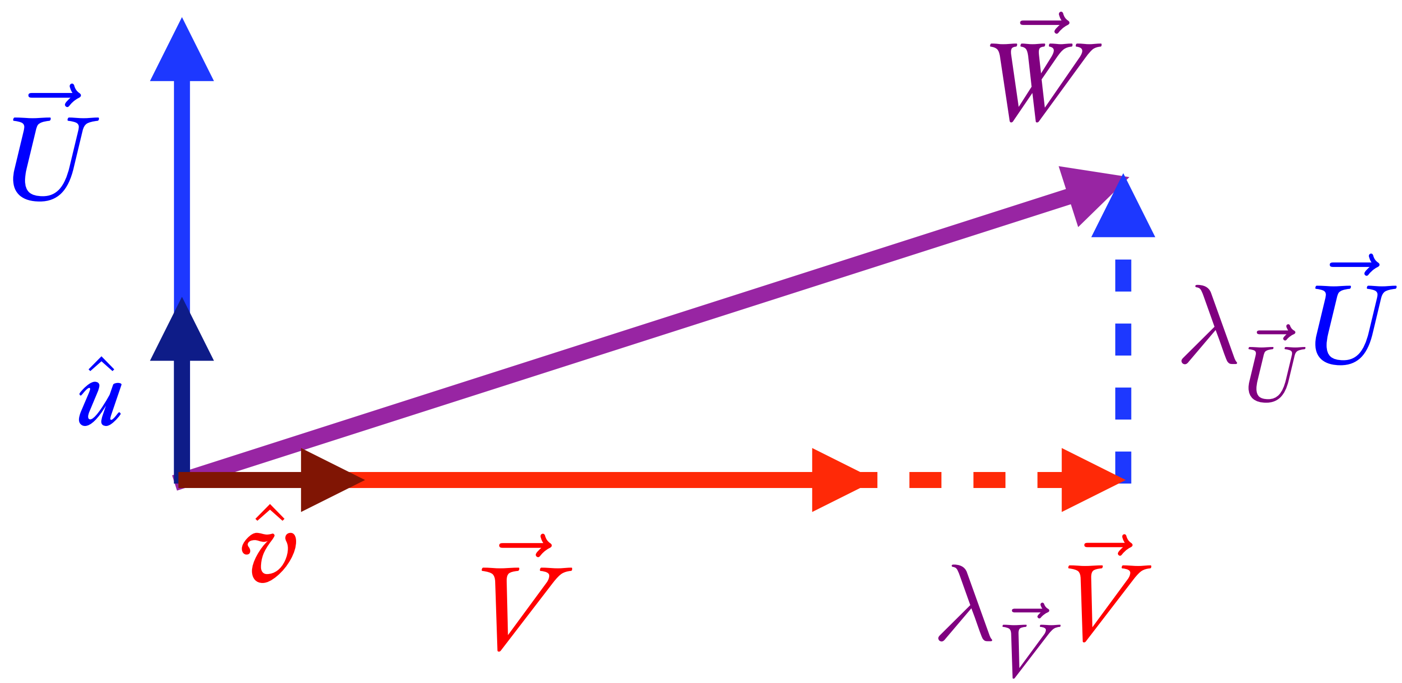
$\hat{u}$ $\hat{v}$



A short reminder on vector algebra

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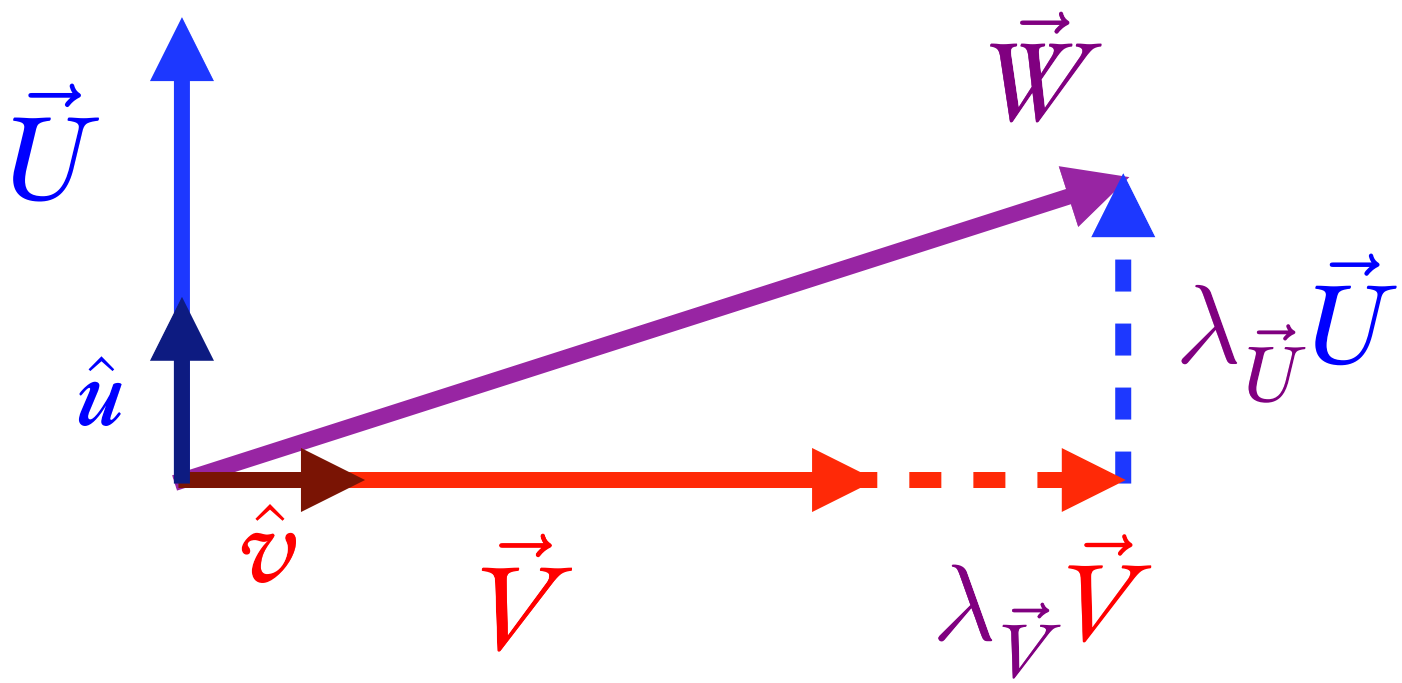


$$\vec{W} = \lambda_{\vec{U}}\vec{U} + \lambda_{\vec{V}}\vec{V}$$

$$\vec{W} = \underbrace{\lambda_{\vec{U}}||\vec{U}||}_{W_{\hat{u}}}\underbrace{\frac{\vec{U}}{||\vec{U}||}}_{\hat{u}} + \underbrace{\lambda_{\vec{V}}||\vec{V}||}_{W_{\hat{v}}}\underbrace{\frac{\vec{V}}{||\vec{V}||}}_{\hat{v}}$$

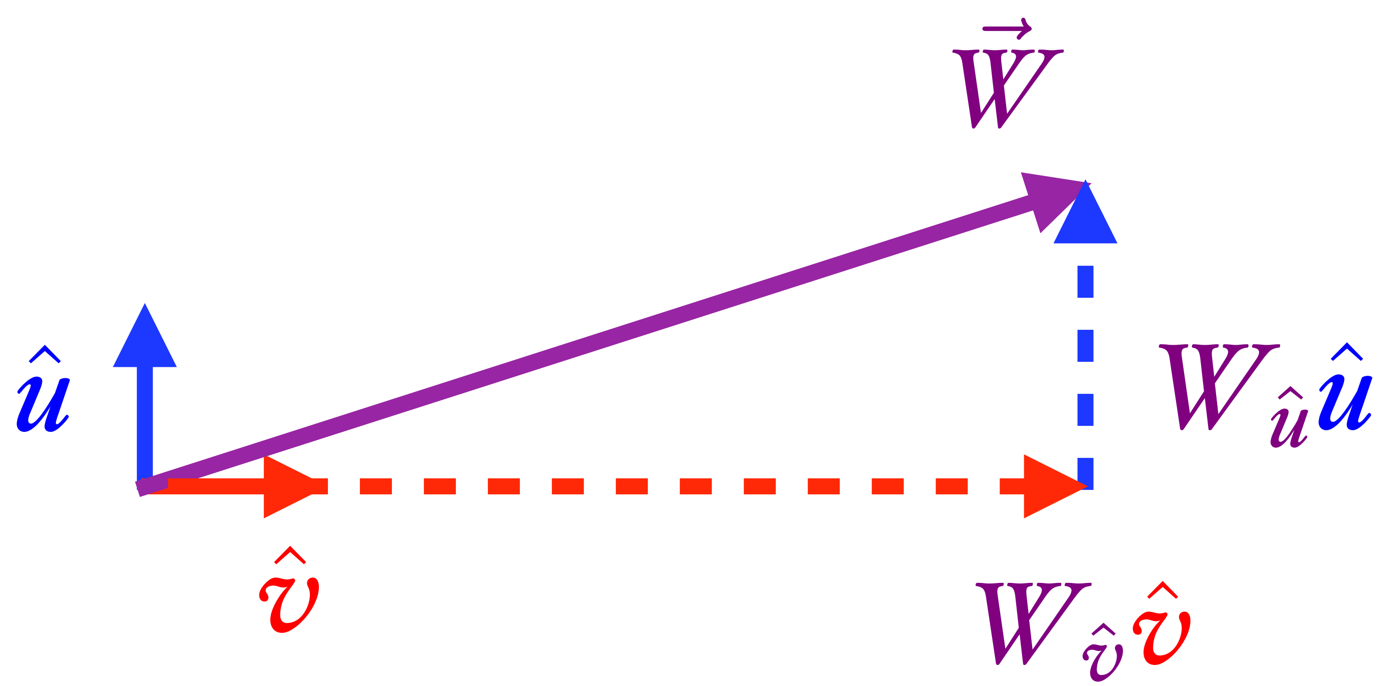
$$\boxed{\vec{W} = W_{\hat{u}}\hat{u} + W_{\hat{v}}\hat{v}}$$

A short reminder on vector algebra

Basis and components

$$\boxed{\vec{W} = W_{\hat{u}}\,\hat{u} + W_{\hat{v}}\,\hat{v}}$$

- The pair $(\hat{u}, \hat{v})$ with $\hat{u} \perp \hat{v}$ and $\|\hat{u}\| = \|\hat{v}\| = 1$, is called an ***orthonormal basis***.
- The numbers $W_{\hat{u}}$ and $W_{\hat{v}}$ are called the ***components*** of $\vec{W}$.



A short reminder on vector algebra

Vector algebra in terms of components

All the vector operations can be implemented with components

- If $\vec{W} = W_{\hat{u}}\,\hat{u} + W_{\hat{v}}\,\hat{v}$ and $\vec{Z} = Z_{\hat{u}}\,\hat{u} + Z_{\hat{v}}\,\hat{v}$

Addition: $$\vec{W} + \vec{Z} = (Z_{\hat{u}} + W_{\hat{u}})\,\hat{u} + (W_{\hat{v}} + Z_{\hat{v}})\,\hat{v}$$

Multiplication by a number: $$\lambda\vec{W} = \lambda W_{\hat{u}}\,\hat{u} + \lambda W_{\hat{v}}\,\hat{v}$$

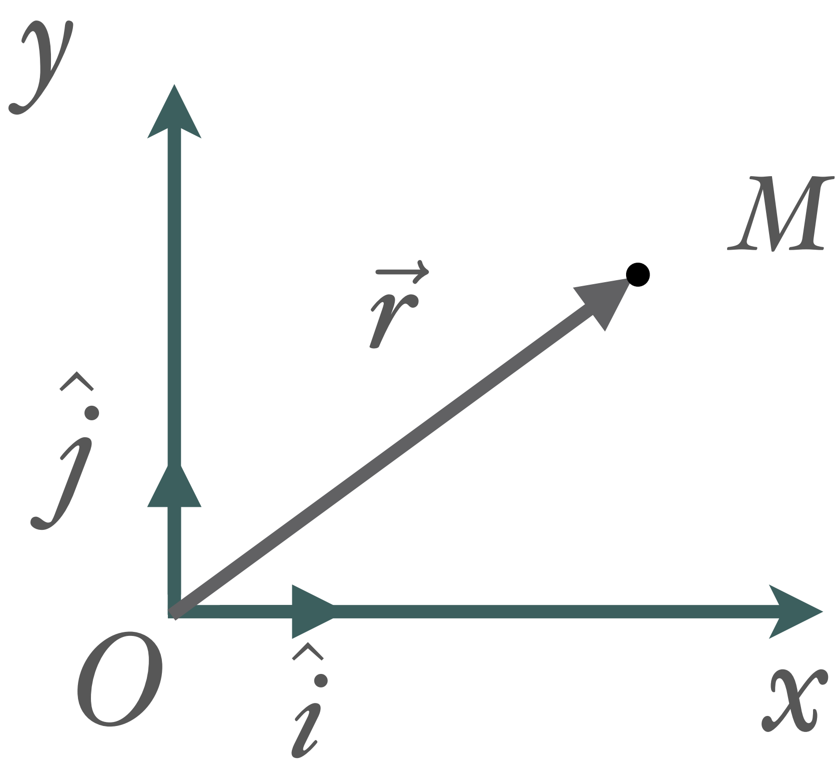
Scalar product: $$\vec{W}\cdot\vec{Z} = W_{\hat{u}}Z_{\hat{u}} + W_{\hat{v}}Z_{\hat{v}}$$

Norm: $$||\vec{W}|| = \sqrt{\vec{W}\cdot\vec{W}} = \sqrt{W_{\hat{u}}{}^2 + W_{\hat{v}}{}^2}$$

Kinematics in 2D

Relative position

Given a ***frame*** consisting of an orthonormal basis $(\hat{i}, \hat{j})$ and an origin O, the position of a point M relative to the origin is characterised by its position vector $\vec{r} = \vec{OM}$ in this frame

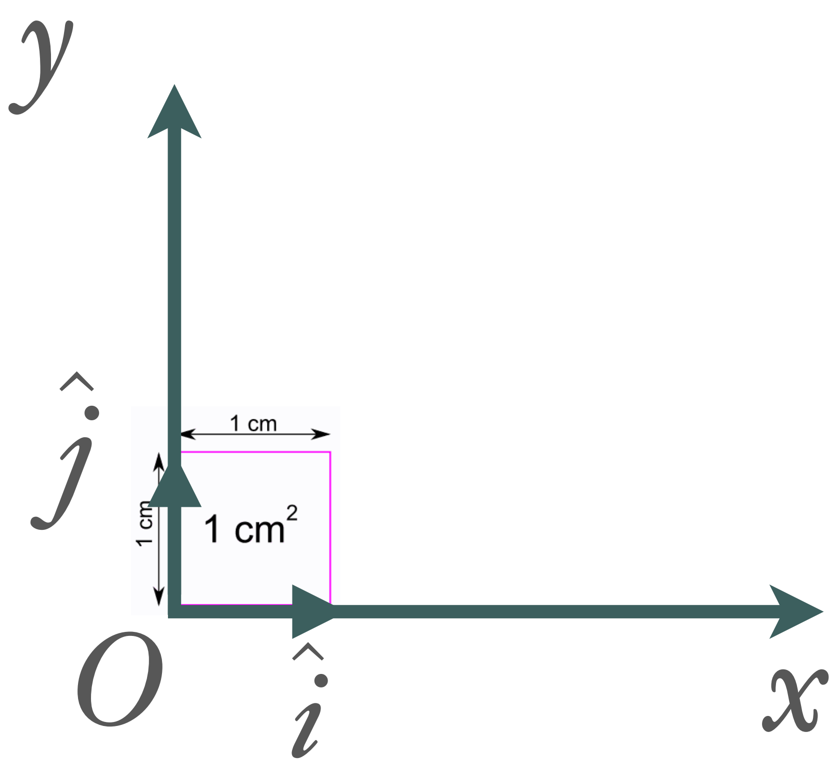


$$\boxed{\vec{r} = x\,\hat{i} + y\,\hat{j}}$$

The components x and y of $\vec{r}$ in the frame $(O, \hat{i}, \hat{j})$ are called the (cartesian) ***coordinates*** of M

Relative position

Example 1



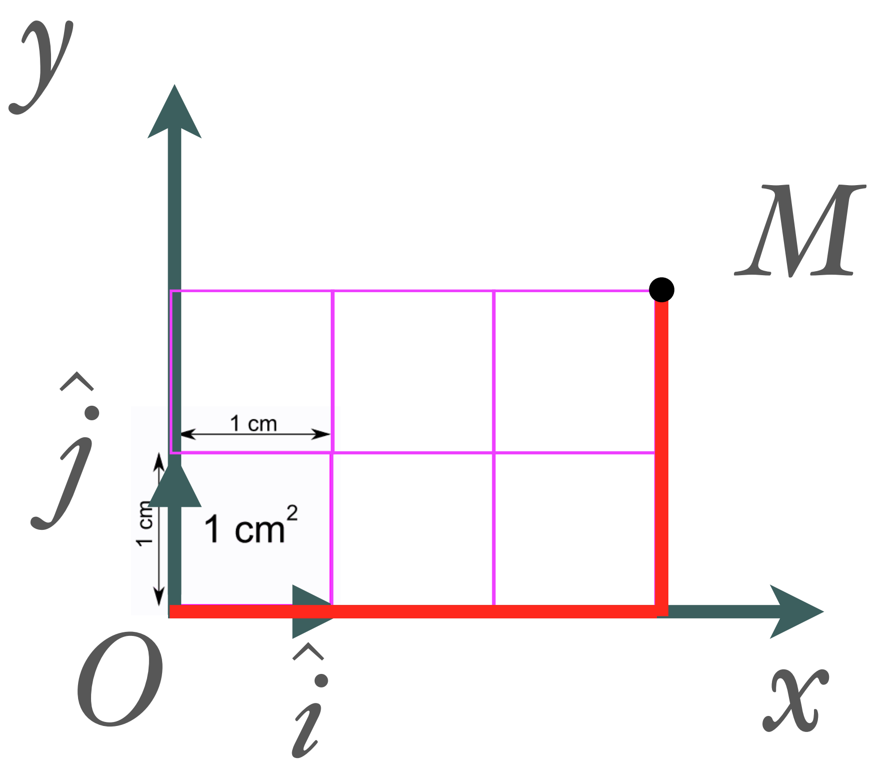
Given the position vector below

$$\vec{r} = (3\ \mathrm{cm})\,\hat{i} + (2\ \mathrm{cm})\,\hat{j}$$

find the position of the corresponding point M on the graph.

Relative position

Example 1



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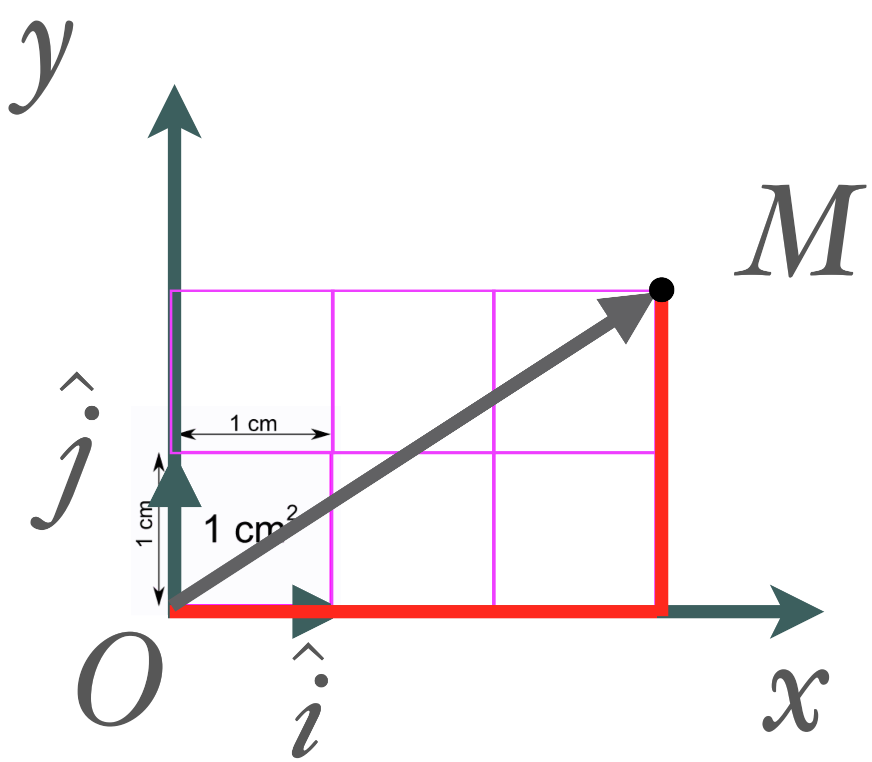
Example 1



Given the position vector below

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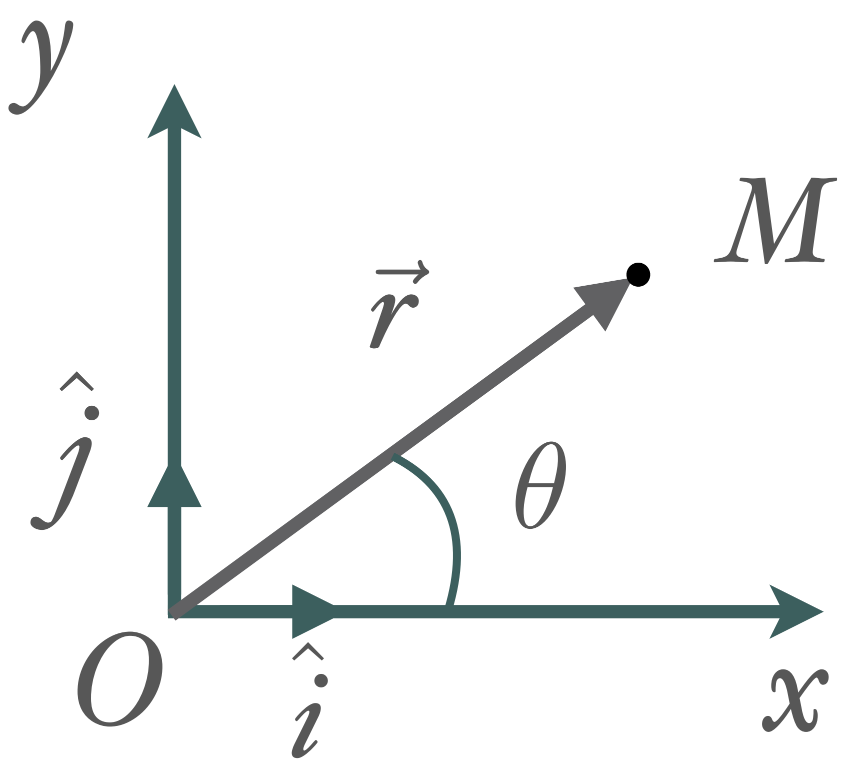
find the position of the corresponding point M on the graph.

Relative position

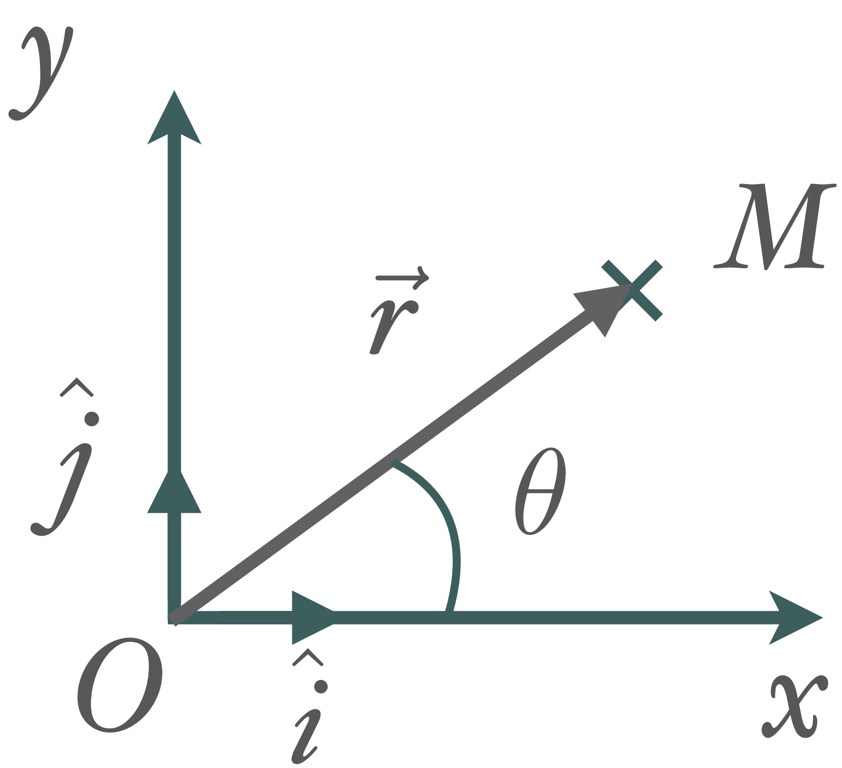
Example 2



Given that $||\vec{r}|| = 4$ cm and $\theta = \pi/4$ rad, find the coordinates of M in the frame $(O, \hat{i}, \hat{j})$.

Relative position

Example 2



Given that $||\vec{r}|| = 4$ cm and $\theta = \pi/4$ rad, find the coordinates of M in the frame $(O, \hat{i}, \hat{j})$.

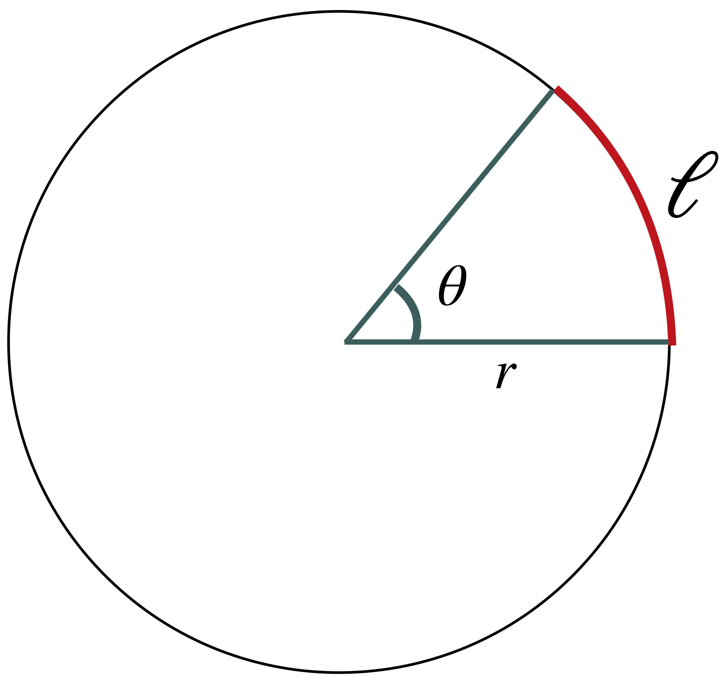
Answer: the coordinates are the components of $\vec{r} = x\hat{i} + y\hat{j}$, with

$$x = \vec{r} \cdot \hat{i} = ||\vec{r}|| \cos\theta = 2\sqrt{2} \text{ cm}$$

$$y = \vec{r} \cdot \hat{j} = ||\vec{r}|| \sin\theta = 2\sqrt{2} \text{ cm}$$

A quick word on the dimension of an angle

The arc length ℓ of a circle is $\ell = r\theta$, where r is the radius of the circle and θ the corresponding angle in **radians**.



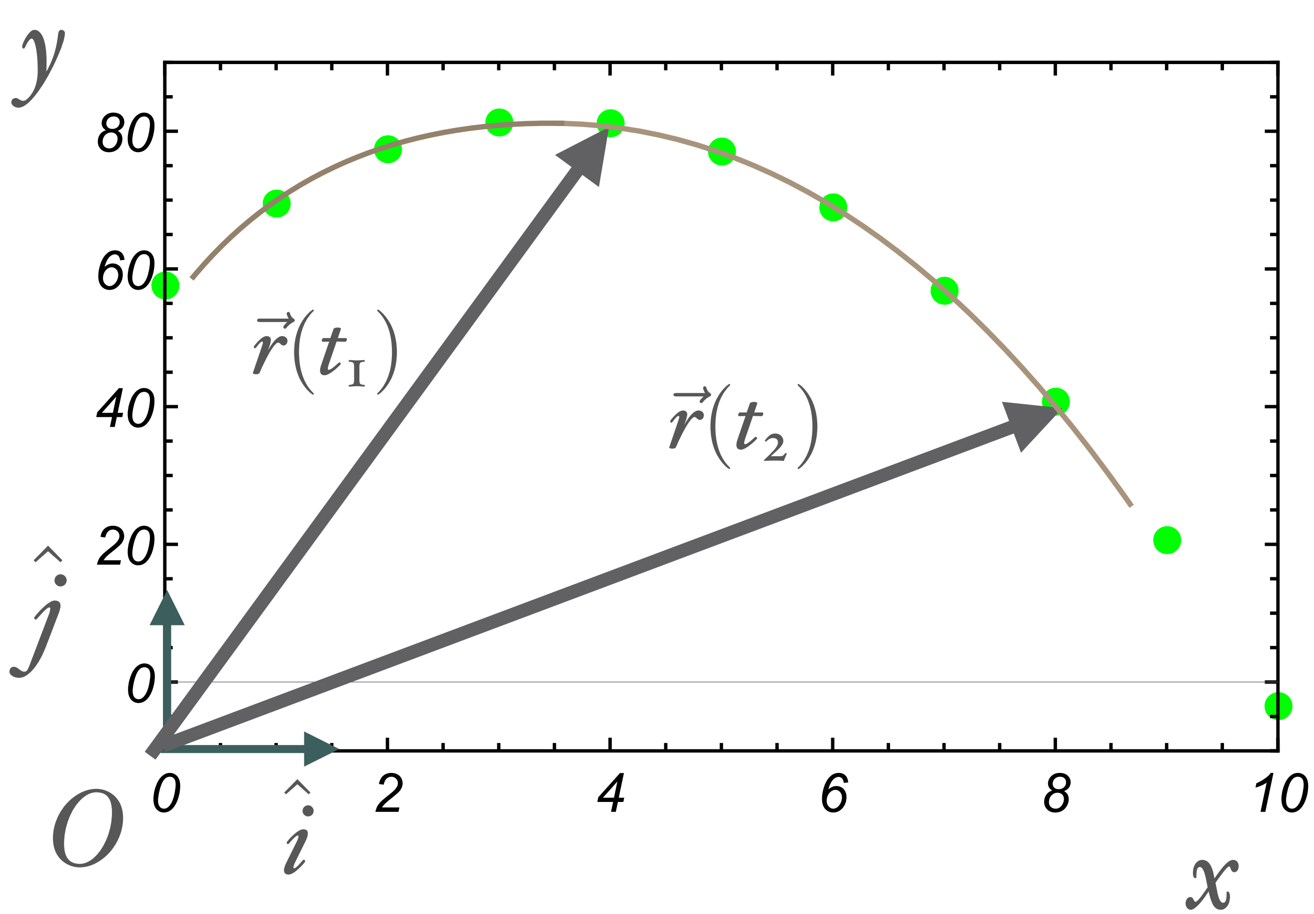
$$\theta = \frac{\ell}{r}. \text{ So, } [\theta] = \frac{[\ell]}{[r]} = \frac{L}{L} = 1,$$

i.e., the angle θ is dimensionless.

Hence, we will consider any angle θ dimensionless and correspondingly $[cos\theta] = [\sin\theta] = 1$.

Average velocity

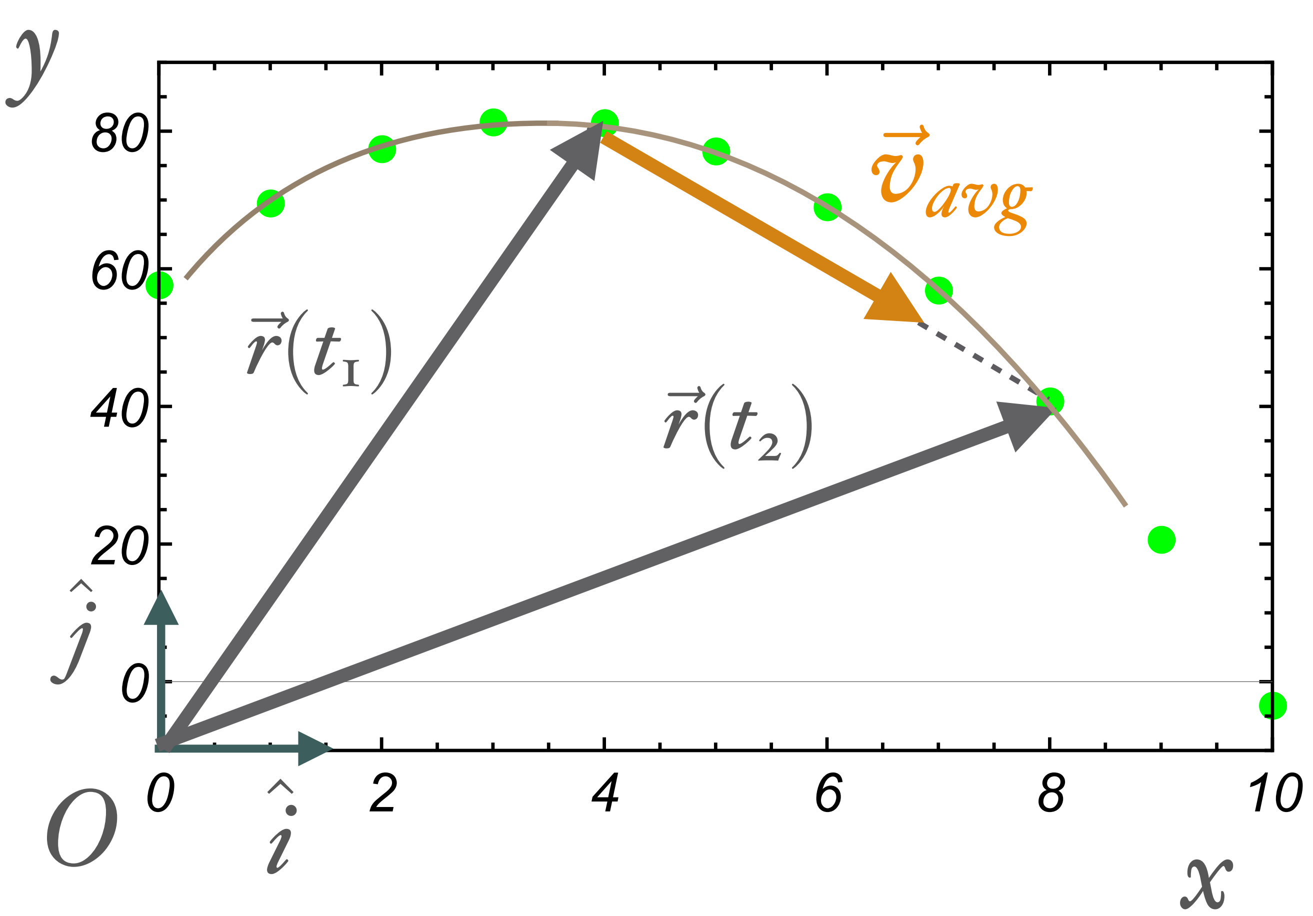
The trajectory of a point object can be represented by its vector position as a function of time $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$.



Average velocity

The trajectory of a point object can be represented by its vector position as a function of time $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$.

The average velocity of point M between t_1 and t_2 is the vector:



$$\vec{v}_{avg} \equiv \frac{\vec{r}(t_2) - \vec{r}(t_1)}{t_2 - t_1}$$

with components:

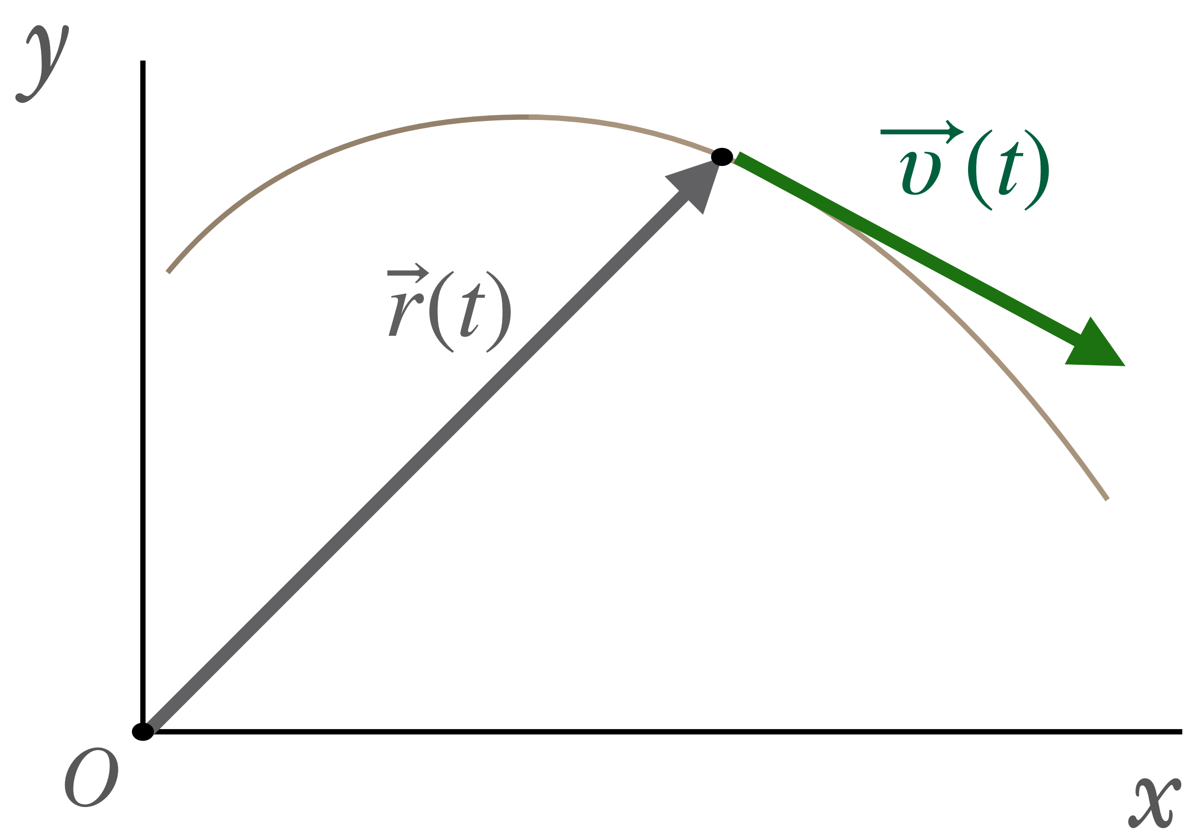
$$v_{x,avg} \equiv \frac{x(t_2) - x(t_1)}{t_2 - t_1}$$

$$v_{y,avg} \equiv \frac{y(t_2) - y(t_1)}{t_2 - t_1}$$

Instantaneous velocity

The instantaneous velocity in 2D is defined as the vector

$$\vec{v}(t) \equiv \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} = \dot{\vec{r}}(t)$$



$$\vec{v}(t) = v_x(t)\,\hat{i} + v_y(t)\,\hat{j}$$

with components:

$$v_x(t) = \dot{x}(t)$$

$$v_y(t) = \dot{y}(t)$$

Instantaneous velocity

Example

We consider the trajectory of a point object represented by the position vector $\vec{r}(t) = (2m)\ \hat{i} - (10\ m/s^2)t^2\ \hat{j}$.
Find the instantaneous velocity at time t .

Instantaneous velocity

Example

We consider the trajectory of a point object represented by the position vector $\vec{r}(t) = (2m)\ \hat{i} - (10\ m/s^2)t^2\ \hat{j}$.
Find the instantaneous velocity at time t.

Solution

$$\overrightarrow{v}(t) = \dot{\vec{r}}(t) = 0 \cdot\ \hat{i} - (20\ m/s^2)\ t\ \hat{j} = -\ (20\ m/s^2)\ t\ \hat{j}.$$

Acceleration

We consider a point object moving with velocity

$$v(t) = v_x(t)\hat{i} + v_y(t)\hat{j}$$

Average acceleration between t_1 and t_2:

$$\overrightarrow{a}_{avg} = \frac{\overrightarrow{v}(t_2) - \overrightarrow{v}(t_1)}{t_2 - t_1} = \frac{v_x(t_2) - v_x(t_1)}{t_2 - t_1}\hat{i} + \frac{v_y(t_2) - v_y(t_1)}{t_2 - t_1}\hat{j}$$

Instantaneous acceleration at t:

$$\overrightarrow{a}(t) = \lim_{h \to 0} \frac{\overrightarrow{v}(t+h) - \overrightarrow{v}(t)}{h} = \dot{\overrightarrow{v}}(t) = \dot{v}_x(t)\hat{i} + \dot{v}_y(t)\hat{j}$$

Instantaneous acceleration

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We consider the trajectory of a point object represented by the position vector $\vec{r}(t) = (2m)\ \hat{i} - (10\ m/s^2)t^2\ \hat{j}$.
Find the instantaneous acceleration at time t .

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Instantaneous acceleration

Example

We consider the trajectory of a point object represented by the position vector $\vec{r}(t) = (2m)\ \hat{i} - (10\ m/s^2)t^2\ \hat{j}$.
Find the instantaneous acceleration at time t .

Solution

$$\overrightarrow{v}(t) = \dot{\vec{r}}(t) = 0 \cdot\ \hat{i} - (20\ m/s^2)\ t\ \hat{j} = -\ (20\ m/s^2)\ t\ \hat{j}.$$

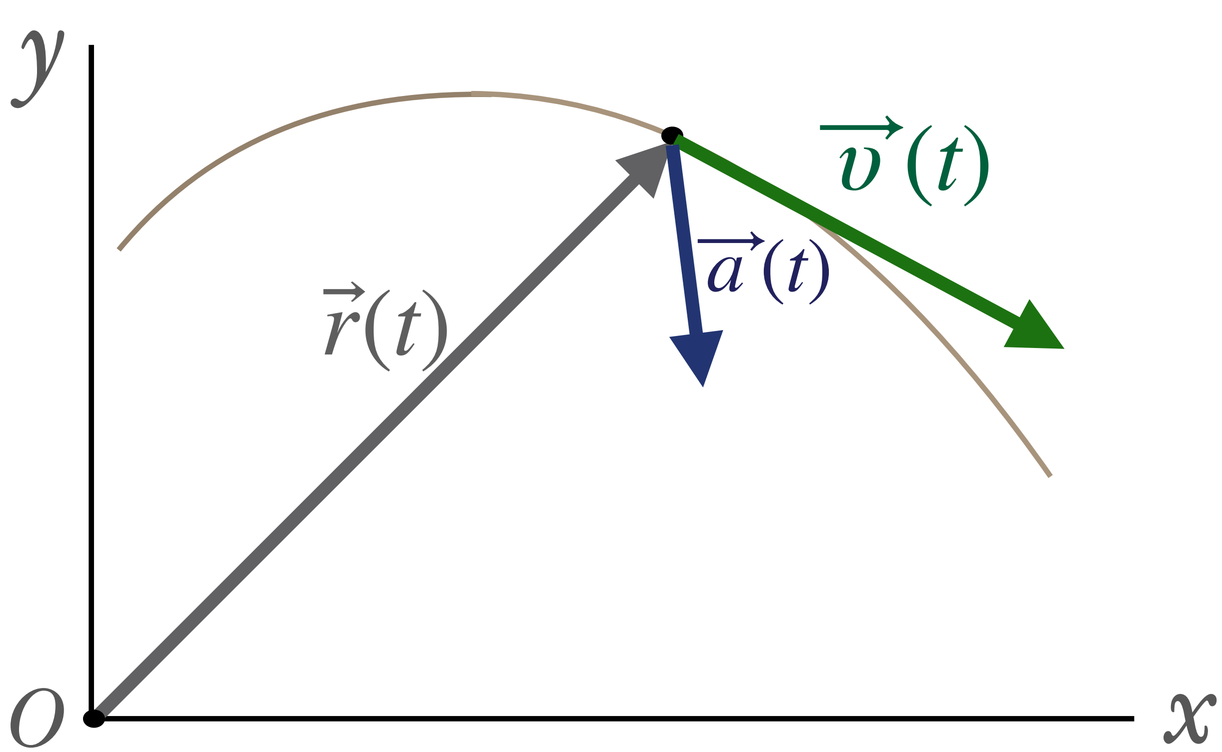
$$\overrightarrow{a}(t) = \dot{\overrightarrow{v}}(t) = -\ (20\ m/s^2)\ \hat{j}.$$

Summary: position, velocity and acceleration

Position $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$

Velocity $\overrightarrow{v}(t) = \dot{\vec{r}}(t) = \dot{x}(t)\hat{i} + \dot{y}(t)\hat{j}$

Acceleration $\overrightarrow{a}(t) = \dot{\overrightarrow{v}}(t) = \ddot{\vec{r}}(t) = \ddot{x}(t)\hat{i} + \ddot{y}(t)\hat{j}$



Uniformly accelerated motion in 2D

The motion of a point object is said to be ***uniformly accelerated*** if at all time during the motion:

$$\boxed{\vec{a}(t) = \vec{a}}$$

where $\overrightarrow{a} = a_x\hat{i} + a_y\hat{j}$ is a constant vector.

Uniformly accelerated motion in 2D

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where $\overrightarrow{a} = a_x\hat{i} + a_y\hat{j}$ is a constant vector.

For $\overrightarrow{a}(t) = a_x(t)\hat{i} + a_y(t)\hat{j}$ we derive

$$\boxed{\begin{aligned} a_x(t) &= a_x \\ a_y(t) &= a_y \end{aligned}}$$

i.e. both components of the acceleration are constants.

Uniformly accelerated motion in 2D

In the case of uniformly accelerated motion we have:

$$a_x(t) = a_x \qquad \longleftrightarrow \qquad \ddot{x}(t) = a_x$$

$$a_y(t) = a_y \qquad \qquad \qquad \ddot{y}(t) = a_y$$

What happens in x-direction is independent of the y-direction

We literally just have to solve twice a 1D problem!

Uniformly accelerated motion in 2D

$$\begin{aligned} a_x(t) &= a_x \\ a_y(t) &= a_y \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \ddot{x}(t) &= a_x \\ \ddot{y}(t) &= a_y \end{aligned}$$

vectors / components	$\vec{v}(t) = v_x(t)\,\hat{i} + v_y(t)\,\hat{j}$	$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j}$
x	$v_x(t) = v_{x,0} + a_x\,t$	$x(t) = x_0 + v_{x,0}\,t + \frac{1}{2}a_x\,t^2$
y	$v_y(t) = v_{y,0} + a_y\,t$	$y(t) = y_0 + v_{y,0}\,t + \frac{1}{2}a_y\,t^2$

The relativity of motion

How is motion relative?

Let us consider the simple example depicted below

Julie (left) and Alice (right) running

How is motion relative?

Let us consider the simple example depicted below

Julie (left) and Alice (right) running

How is motion relative?

Let us consider the simple example depicted below

Julie (left) and Alice (right) running

* Although they are running Alice and Julie are not moving with respect to each other

* They nevertheless move with respect to the stop sign

Relative motion in equation

* How could we express the fact that Alice and Julie do not move relative to each other?

  Introduce the point objects A for Alice and J for Julie. It then follows that

$$\boxed{\overrightarrow{\mathrm{AJ}} = \textit{constant} \text{ or } \dot{\overrightarrow{\mathrm{AJ}}} = 0}$$

Relative motion in equation

* How could we express the fact that Alice and Julie do not move relative to each other?

  Introduce the point objects A for Alice and J for Julie. It then follows that

$$\boxed{\overrightarrow{\mathrm{AJ}} = \textit{constant} \text{ or } \dot{\overrightarrow{\mathrm{AJ}}} = 0}$$

* How could we express the fact that Alice and Julie move however with respect to the stop sign?

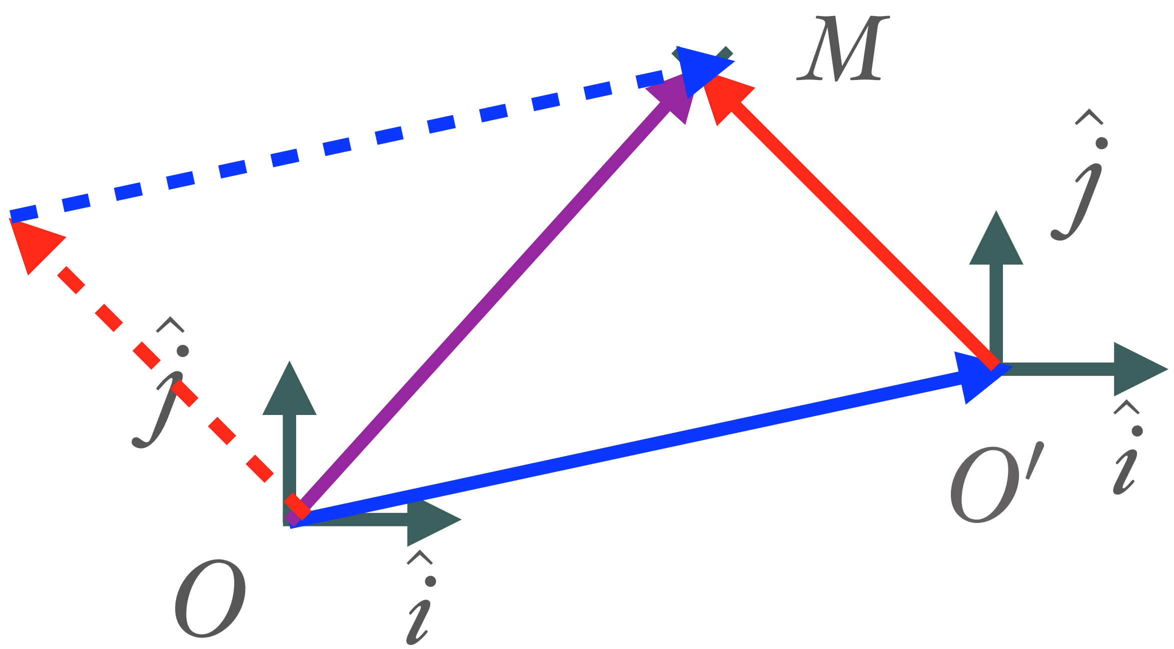
Introduce the point object S for the stop sign. It then follows that

$$\boxed{\dot{\overrightarrow{\mathrm{SA}}} \neq 0 \text{ and } \dot{\overrightarrow{\mathrm{SJ}}} \neq 0}$$

Relative motion for frames in translation

We consider two frames $\mathcal{F} = (O,\, \hat{i},\, \hat{j})$ and $\mathcal{F}\,' = (O',\, \hat{i},\, \hat{j})$ and the point object M

$$\boxed{\overrightarrow{OM} = \overrightarrow{OO'} + \overrightarrow{O'M}}$$



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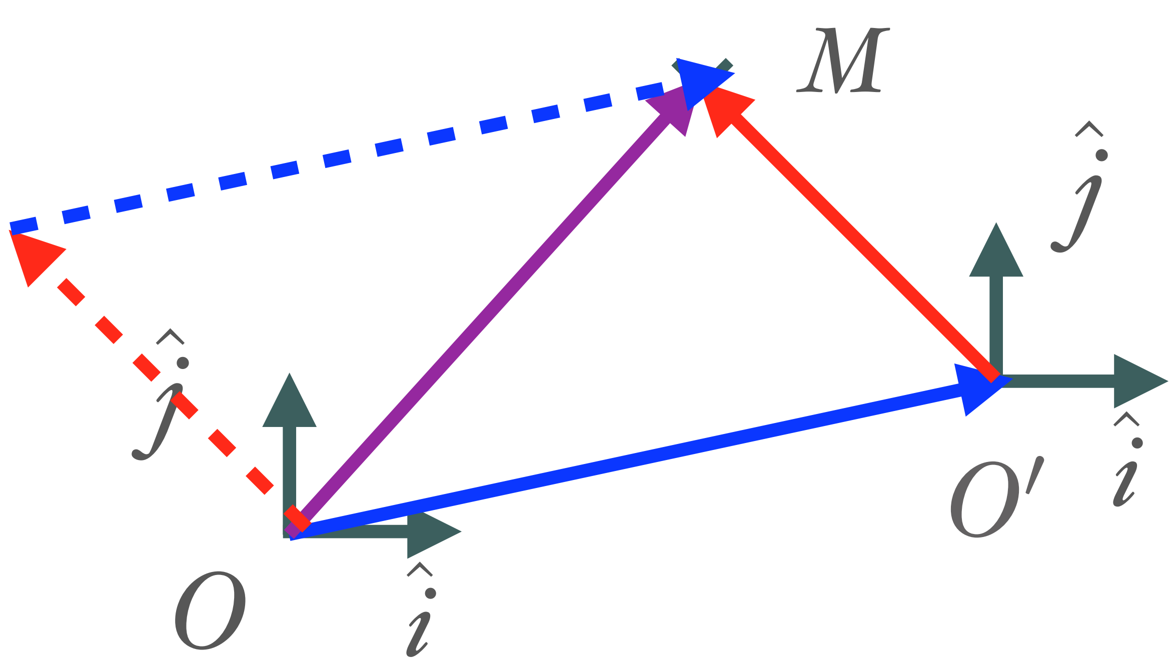
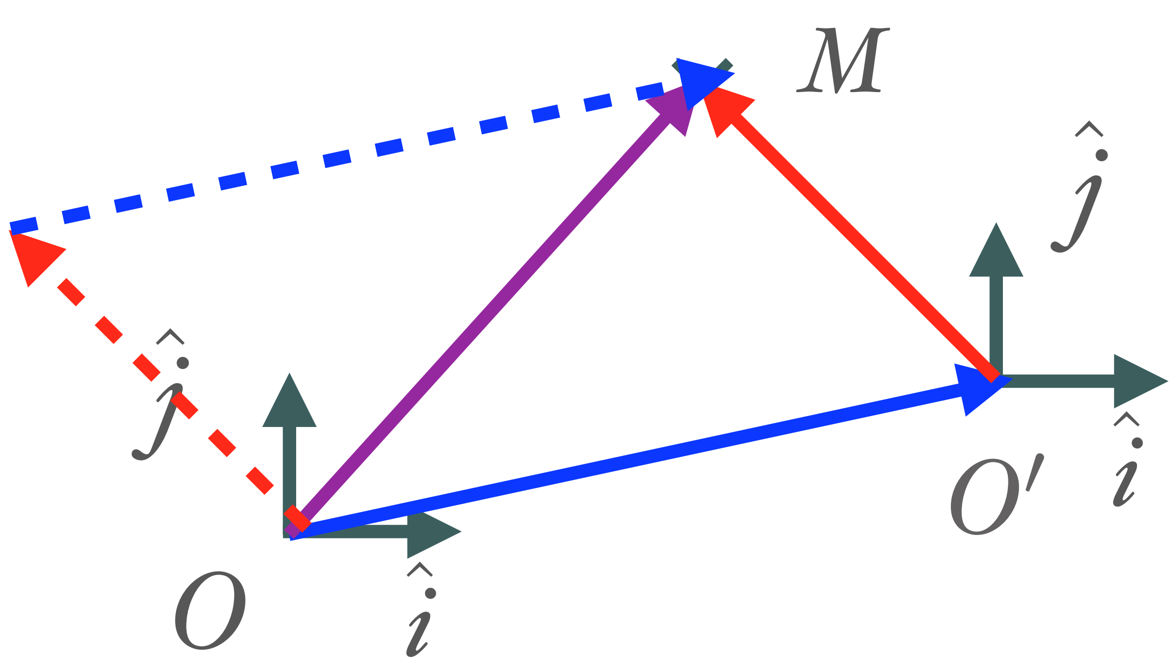


The velocity and the acceleration of a point M relative to O is defined as

$$\vec{v}(M|O) \equiv \dot{\overrightarrow{OM}}$$
$$\vec{a}(M|O) \equiv \dot{\vec{v}}(M|O) = \ddot{\overrightarrow{OM}}$$

Relative motion for frames in translation

We consider two frames $\mathcal{F} = (O, \hat{i}, \hat{j})$ and $\mathcal{F}' = (O', \hat{i}, \hat{j})$ and the point object M



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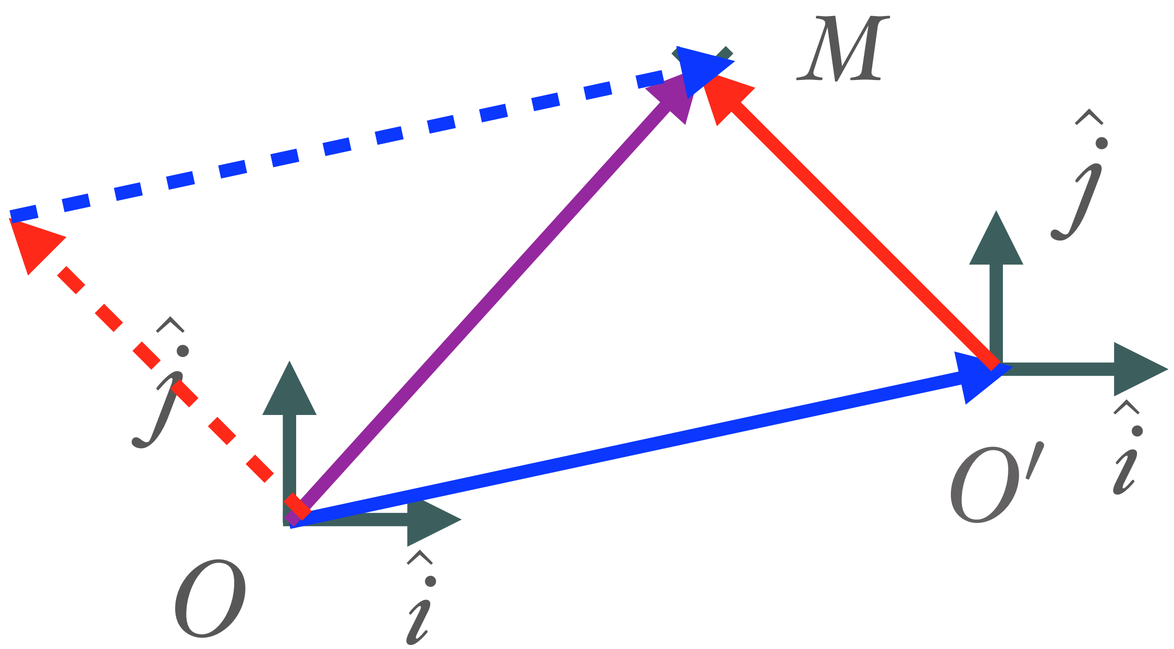
$$\dot{\overrightarrow{OM}} = \dot{\overrightarrow{OO'}} + \dot{\overrightarrow{O'M}} \longrightarrow \overrightarrow{v}(M|O) = \overrightarrow{v}(O'|O) + \overrightarrow{v}(M|O')$$

$$\ddot{\overrightarrow{OM}} = \ddot{\overrightarrow{OO'}} + \ddot{\overrightarrow{O'M}} \longrightarrow \overrightarrow{a}(M|O) = \overrightarrow{a}(O'|O) + \overrightarrow{a}(M|O')$$

Relative motion for frames in translation



Law of composition of velocities $\vec{v}(M|O) = \vec{v}(O'|O) + \vec{v}(M|O')$

Law of composition of accelerations $\vec{a}(M|O) = \vec{a}(O'|O) + \vec{a}(M|O')$

Relative motion

Example

Julie (left) and Alice (right) running

Let us suppose that the velocity of Julie relative to the stop is

$$\vec{v}(\mathrm{J}|\mathrm{S}) = (7\ \mathrm{km}\cdot\mathrm{h}^{-1})\ \hat{\mathrm{i}}$$

and that the velocity of Alice relative to Julie is zero.

Find the velocity and the acceleration of Alice relative to the stop.

Relative motion

Example

Julie (left) and Alice (right) running

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and that the velocity of Alice relative to Julie is zero.

Find the velocity and the acceleration of Alice relative to the stop.

Solution

$$\vec{v}(\mathrm{A}|\mathrm{S}) = \vec{v}(\mathrm{A}|\mathrm{J}) + \vec{v}(\mathrm{J}|\mathrm{S}) = (7\ \mathrm{km} \cdot \mathrm{h}^{-1})\ \hat{i}$$

$$\vec{a}(\mathrm{A}\,|\,S) = \dot{\vec{v}}(\mathrm{A}\,|\,S) = 0$$