MTH1001-Algebra

Slides Week 8

Symmetric polynomials in two indeterminates.
More examples of symmetric systems.
Symmetric functions of the roots of a cubic polynomial.

Symmetric polynomials

- A polynomial $f(x, y)$ in x and y is *symmetric* if $f(y, x) = f(x, y)$.
- Example: $x^3 + 2x^2y + 2xy^2 + y^3 + 5xy - 4x - 4y + 7$.
- The polynomials $x + y$ and xy are the *elementary symmetric polynomials* in x and y.
- THEOREM. *Every symmetric polynomial $f(x, y)$ can be written as a polynomial in the elementary symmetric polynomials $x + y$ and xy.*
- This means that if $f(x, y)$ is symmetric, then $f(x, y) = g(x + y, xy)$ for some polynomial $g(s, p)$. Here $s = x + y$ and $p = xy$.
- EXAMPLES. $x^2y^3 + x^3y^2 = (xy)^2(x + y) = sp^2$
- $x^2 + y^2 = (x + y)^2 - 2xy = s^2 - 2p$

- $x^3 + y^3 = (x+y)^3 - 3xy(x+y) = s^3 - 3sp$
- $x^4 + y^4 = (x+y)^4 - xy(4x^2 + 6xy + 4y^2)$
  $= (x+y)^4 - xy\big(4(x+y)^2 - 2xy\big)$
  $= (x+y)^4 - 4xy(x+y)^2 + 2(xy)^2 = s^4 - 4s^2p + 2p^2$
- One can continue with $x^5 + y^5$, etc.: sums $x^n + y^n$ of higher powers can also be expressed as polynomials in $x + y$ and xy.
- EXAMPLE. $f(x,y) = x^3 + 2x^2y + 2xy^2 + y^3 + 5xy - 4x - 4y + 7$, a symmetric polynomial, can be written as

  $$f(x,y) = (x+y)^3 - xy(x+y) + 5xy - 4(x+y) + 7 = g(x+y, xy),$$

  where $g(s,p) = s^3 - sp - 4s + 5p + 7$.

More examples of symmetric systems

- EXAMPLE. Solve $\begin{cases} x^4 + y^4 = 17 \\ x + y = 3 \end{cases}$
  - Because $x^4 + y^4 = (x+y)^4 - 4xy(x+y)^2 + 2(xy)^2$, and $x + y = 3$,

$$\begin{cases} 3^4 - 4 \cdot 3^2 \cdot xy + 2(xy)^2 = 17 \\ x + y = 3 \end{cases}$$

  - $$\begin{cases} (xy)^2 - 18(xy) + 32 = 0 \\ x + y = 3 \end{cases}$$
  - Solving the first equation for xy, the system is equivalent to

$$\begin{cases} xy = 2 \\ x + y = 3 \end{cases} \quad \text{or} \quad \begin{cases} xy = 16 \\ x + y = 3 \end{cases}$$

  - Solving these two systems as we learnt earlier we find

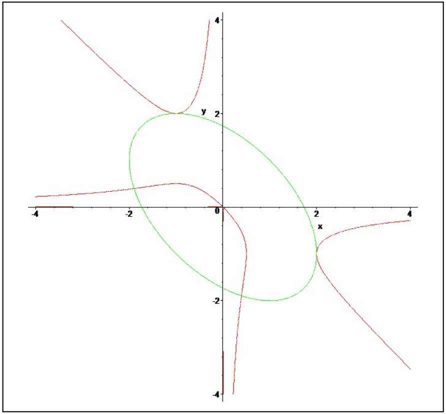
$$(x, y) = (2, 1), (1, 2), \left(\frac{3}{2} \pm i\frac{\sqrt{55}}{2}, \frac{3}{2} \mp i\frac{\sqrt{55}}{2} \right).$$

- EXAMPLE. Solve $\begin{cases} x + y + \frac{1}{x} + \frac{1}{y} &= \frac{1}{2} \\ x^2 + y^2 + xy &= 3 \end{cases}$ Multiply by $2xy$, get
  - $\begin{cases} 2(x+y)xy + 2(x+y) &= xy \\ (x+y)^2 - xy &= 3 \end{cases}$ a system of degree $3 \cdot 2 = 6$.
  - Get xy from the second equation and substitute into the first, so $\begin{cases} 2(x+y)^3 - 4(x+y) &= (x+y)^2 - 3 \\ xy &= (x+y)^2 - 3 \end{cases}$
  - Setting $s = x + y$ the first equation becomes $2s^3 - s^2 - 4s + 3 = 0$.
  - Using the Rational Root Test we find the roots 1 (twice), and $-3/2$, so the first equation is $(s-1)^2(2s+3) = 0$.
  - For each of those values of $s = x + y$ the other equation of the system will give $xy = s^2 - 3$, so the system is equivalent to

$$\begin{cases} x + y = 1 \\ xy = -2 \end{cases} \quad \text{or} \quad \begin{cases} x + y = -3/2 \\ xy = -3/4 \end{cases}$$

(where each root of the first system should really be counted twice).

- Solving the two systems we find four solutions (all real) $(x, y) = (2, -1),\ (-1, 2),\ \left(\frac{-3+\sqrt{21}}{2}, \frac{-3-\sqrt{21}}{2}\right),\ \left(\frac{-3+\sqrt{21}}{2}, \frac{-3+\sqrt{21}}{2}\right).$
- The first two should actually be counted as *double solutions.* Counting that way we have found six solutions of our system.
- The double roots are the points in the graph where the two curves (red and green) intersect with the same tangent.



- EXAMPLE. Compute $\alpha^2+\beta^2$, $\alpha^3+\beta^3$, and $\alpha^{-2}+\beta^{-2}$, where α and β are the roots of the polynomial x^2-5x+3.
  - Computing α and β first, which are $(5\pm\sqrt{13})/2$, would involve calculations with radicals, which would eventually simplify.
  - It is much better to use the fact that $\alpha+\beta=5$ and $\alpha\beta=3$, whence

$$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=5^2-2\cdot 3=19,$$

$$\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)=5^3-3\cdot 3\cdot 5=80.$$

$$\alpha^{-2}+\beta^{-2}=\frac{\alpha^2+\beta^2}{(\alpha\beta)^2}=\frac{(\alpha+\beta)^2-2\alpha\beta}{(\alpha\beta)^2}=\frac{5^2-2\cdot 3}{3^2}=\frac{19}{9}.$$

  - This last one could also be found by looking at the reciprocal polynomial $3x^2-5x+1$, whose roots are $1/\alpha$ and $1/\beta$:

$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}=\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)^2-2\cdot\frac{1}{\alpha}\cdot\frac{1}{\beta}=\left(\frac{5}{3}\right)^2-2\cdot\frac{1}{3}=\frac{19}{9}.$$

Symmetric functions of the roots of a cubic polynomial

- Symmetric functions of the roots generalize to polynomials of higher degree. Look at the case of a cubic polynomial.
- We may assume it monic and write it as $x^3 - sx^2 + rx - p$. Then

$$x^3 - sx^2 + rx - p = (x-\alpha)(x-\beta)(x-\gamma)$$
$$= x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta+\alpha\gamma+\beta\gamma)x - \alpha\beta\gamma,$$

where α, β, γ are its roots in some field, and so

$$\alpha+\beta+\gamma = s, \qquad \alpha\beta+\alpha\gamma+\beta\gamma = r, \qquad \alpha\beta\gamma = p.$$

- The LHS are the *elementary symmetric polynomials* in α, β, γ.
- A polynomial $f(\alpha, \beta, \gamma)$ in three indeterminates α, β, γ is *symmetric* if it is unchanged after *permuting* α, β, γ in all (six) possible ways.
- THEOREM. *Every symmetric polynomial $f(\alpha, \beta, \gamma)$ can be written as a polynomial in the elementary symmetric polynomials* $\alpha+\beta+\gamma$, $\alpha\beta+\alpha\gamma+\beta\gamma$, *and* $\alpha\beta\gamma$.

- EXAMPLE. Express $\alpha^2\beta + \alpha^2\gamma + \beta^2\gamma + \alpha\beta^2 + \alpha\gamma^2 + \beta\gamma^2$ in terms of the elementary symmetric polynomials in α, β, γ.
  - First of all, note that this is actually a symmetric polynomial: take $\alpha^2\beta$, apply all permutations of α, β, γ, and you get precisely all terms of the sum (and each exactly once in this case).
  - By the Theorem it must be possible to express it as a polynomial in
  $$s = \alpha + \beta + \gamma, \qquad r = \alpha\beta + \alpha\gamma + \beta\gamma, \qquad p = \alpha\beta\gamma.$$
  - Because $\alpha^2\beta = (\alpha\beta) \cdot \alpha$, this suggests considering the product rs:
  $$\begin{aligned}(\alpha\beta + \alpha\gamma + \beta\gamma)(\alpha + \beta + \gamma) &= \alpha^2\beta + \alpha^2\gamma + \alpha\beta\gamma \\ &\quad + \alpha\beta^2 + \alpha\beta\gamma + \beta^2\gamma \\ &\quad + \alpha\beta\gamma + \alpha\gamma^2 + \beta\gamma^2.\end{aligned}$$
  - Consequently, $\alpha^2\beta + \alpha^2\gamma + \beta^2\gamma + \alpha\beta^2 + \alpha\gamma^2 + \beta\gamma^2$
  $$= (\alpha\beta + \alpha\gamma + \beta\gamma)(\alpha + \beta + \gamma) - 3\alpha\beta\gamma = rs - 3p.$$

- EXAMPLE. Let α, β, γ be the complex roots of $x^3 + x^2 - 2x - 5$.
  - Then $\alpha + \beta + \gamma = -1, \quad \alpha\beta + \alpha\gamma + \beta\gamma = -2, \quad \alpha\beta\gamma = 5$.
  - $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma) = (-1)^2 - 2 \cdot (-2) = 5$.
  - $$\begin{aligned} \alpha^{-2} + \beta^{-2} + \gamma^{-2} = \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} &= \frac{\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2}{(\alpha\beta\gamma)^2} \\ &= \frac{(\alpha\beta + \alpha\gamma + \beta\gamma)^2 - 2(\alpha\beta\gamma)(\alpha + \beta + \gamma)}{(\alpha\beta\gamma)^2} \\ &= \frac{(-2)^2 - 2 \cdot 5 \cdot (-1)}{5^2} = \frac{14}{25}. \end{aligned}$$
  - Or look at $-5x^3 - 2x^2 + x + 1$, whose roots are $\alpha^{-1}, \beta^{-1}, \gamma^{-1}$, so
    $$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = -\frac{2}{5}, \qquad \frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma} = -\frac{1}{5}, \qquad \frac{1}{\alpha\beta\gamma} = \frac{1}{5},$$
  - and hence $$\begin{aligned} \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} &= \left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right)^2 - 2\left(\frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma}\right) \\ &= \left(-\frac{2}{5}\right)^2 - 2 \cdot \left(-\frac{1}{5}\right) = \frac{14}{25}. \end{aligned}$$