MTH1001-Algebra

Slides Week 6

The rational root test.
Irreducibility and roots of quadratic and cubic polynomials.
Biquadratic polynomials.
Self-reciprocal polynomials.
Square roots of complex numbers.
Relations between roots and coefficients of a quadratic polynomial.
Symmetric systems.

The rational root test

- This test is used to find *all* rational roots of a polynomials with rational coefficients, or to conclude that none exist if none is found.
- Multiplying by a suitable scalar we get a polynomial in $\mathbb{Z}[x]$.
- THEOREM. *Let* $f(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$*.*
  If r/s *is a rational root of* $f(x)$*, with* $r, s \in \mathbb{Z}$ *and* $(r, s) = 1$*, then* r *divides* a_0*, and* s *divides* a_n*.*
- PROOF. Expand $s^n \cdot f(r/s) = 0$:
  - $a_n r^n + a_{n-1} r^{n-1} s + a_{n-2} r^{n-2} s^2 + \cdots + a_2 r^2 s^{n-2} + a_1 r s^{n-1} + a_0 s^n = 0.$
  - r divides all terms preceding the last one, so $r \mid a_0 s^n$ as well.
  - Because $(r, s) = 1$, Arithmetical Lemma B implies that r divides a_0.
  - Similarly, s divides all terms following the first one, so $s \mid a_n r^n$ as well. Because $(r, s) = 1$ it follows that s divides a_n. $\square$

- EXAMPLE. Find all rational roots of $f(x) = 2x^3 + 15x^2 + 27x + 10$.
  - According to the test, if $r/s \in \mathbb{Q}$ is a root of $f(x)$, with $(r, s) = 1$, then r divides 10 and s divides 2, hence $r \in \{\pm 1, \pm 2, \pm 5, \pm 10\}$ and $s \in \{\pm 1, \pm 2\}$. Consequently, the possibilities for r/s are

$$\pm 1,\ \pm 2,\ \pm 5,\ \pm 10,\ \pm 1/2,\ \pm 5/2.$$

  - Because the coefficients of the polynomial are all > 0, no real number > 0 can be a root, so here only those < 0 need testing.
  - Going through the list we find $f(-2) = f(-5) = f(-1/2) = 0$, then we can stop (as there cannot be more than three roots), and

$$f(x) = 2(x+2)(x+5)(x+\tfrac{1}{2}) = (x+2)(x+5)(2x+1).$$

  - This is the complete factorisation of $f(x)$ in $\mathbb{Q}[x]$, and in the last form it is actually in $\mathbb{Z}[x]$. (This can always be done in $\mathbb{Z}[x]$, if $f(x) \in \mathbb{Z}[x]$, by rearranging some scalar factors; known as *Gauss's Lemma*.)
  - Alternatively, after finding the first root -2 we may divide $f(x)$ by $x + 2$, and then factorise the resulting quadratic polynomial.

Application: irrationality of certain radicals

- EXAMPLE. We prove that $\sqrt{3}$ is irrational.
  - $\sqrt{3}$ is a root of the polynomial $x^2 - 3$.
  - By the Rational Root Test, if r/s is a rational root of $x^2 - 3$, with $r, s \in \mathbb{Z}$ and $(r, s) = 1$, then $r \mid 3$ and $s \mid 1$, and so $r/s \in \{\pm 1, \pm 3\}$.
  - None of those is a root, hence $x^2 - 3$ has no rational root, and so $\sqrt{3}$ is irrational.
- EXAMPLE. We prove that $\sqrt[3]{25/3}$ is irrational.
  - $\sqrt[3]{25/3}$ is a root of the polynomial $3x^3 - 25$.
  - By the Rational Root Test, if r/s is a rational root of $3x^3 - 25$, with $r, s \in \mathbb{Z}$ and $(r, s) = 1$, then $r \mid 25$ and $s \mid 3$, and so $r/s \in \{\pm 1, \pm 5, \pm 25, \pm 1/3, \pm 5/3, \pm 25/3\}$.
  - Because $3 \cdot 2^3 - 25 = -1 < 0$, and $3 \cdot 3^3 - 25 = 56 > 0$, and $x \mapsto x^3$ is increasing, any real root of $3x^3 - 25$ must be > 2 and < 3.
  - None of the possibilities is > 2 and < 3, hence $3x^2 - 25$ has no rational root, and so $\sqrt[3]{25/3}$ is irrational.

Irreducibility of quadratic and cubic polynomials

- PROPOSITION. *If a quadratic or cubic polynomial in $F[x]$ has no root in F, then it is irreducible in $F[x]$.*
- PROOF. If it were reducible, at least one of its factors would have degree one, say $ax + b$, and then $-b/a$ would be a root in F. $\square$
- EXAMPLE. Consider $x^3 - 2$ in $\mathbb{Q}[x]$, or in $\mathbb{R}[x]$, or in $\mathbb{C}[x]$.
  - In $\mathbb{R}[x]$ it is reducible, and its complete factorisation is

    $$x^3 - 2 = (x - \sqrt[3]{2})(x^2 + \sqrt[3]{2}\,x + \sqrt[3]{2}^2).$$

    In fact, the quadratic factor is irreducible over $\mathbb{R}$, having no real root.
  - In $\mathbb{C}[x]$ its complete factorisation is

    $$x^3 - 2 = (x - \sqrt[3]{2})(x - \omega\sqrt[3]{2})(x - \bar{\omega}\sqrt[3]{2}),$$

    where $\omega = (-1 \pm i\sqrt{3})/2$.
  - $\sqrt[3]{2}$ is irrational (Rational Root test), the other two roots are not even real, so $x^3 - 2$ has no root in $\mathbb{Q}$. Hence $x^3 - 2$ is irreducible in $\mathbb{Q}[x]$.

- Polynomials of degree > 3 may be reducible but have no root in F.
- EXAMPLE. $x^4 + 5x^2 + 4 = (x^2 + 1)(x^2 + 4)$ in $\mathbb{R}[x]$, reducible but without real roots. (Its complex roots are $\pm i$ and $\pm 2i$.)
- EXAMPLE. $x^4 + 1$ has no roots in $\mathbb{R}$, but is not irreducible in $\mathbb{R}[x]$:

$$\begin{aligned} x^4 + a^4 &= (x^4 + 2a^2x^2 + a^4) - 2a^2x^2 \\ &= (x^2 + a^2)^2 - (\sqrt{2}ax)^2 \\ &= [(x^2 + a^2) - \sqrt{2}ax][(x^2 + a^2) + \sqrt{2}ax] \\ &= (x^2 - \sqrt{2}ax + a^2)(x^2 + \sqrt{2}ax + a^2). \end{aligned}$$

- Hence $x^4 + a^4$ has no real roots, if $a \in \mathbb{R}$ and $a \neq 0$, but is reducible in $\mathbb{R}[x]$ (complete factorisation given above).
- For example, $x^4 + 4 = (x^2 - 2x + 2)(x^2 + 2x + 2)$, even in $\mathbb{Q}[x]$.
- But $x^4 + 1 = (x^2 - \sqrt{2}x + 1)(x^2 + \sqrt{2}x + 1)$, in $\mathbb{R}[x]$ but not $\mathbb{Q}[x]$.

Biquadratic polynomials

- A *biquadratic polynomial* has the form $ax^4 + bx^2 + c$, with $a \neq 0$.
- To find its roots set $y = x^2$, and then solve $ay^2 + by + c = 0$.
- If β_1, β_2 are the roots of this equation in y, then the roots of $ax^4 + bx^2 + c$ are the solutions of $x^2 = \beta_1$ and those of $x^2 = \beta_2$.
- EXAMPLE. Given $2x^4 + x^2 - 6$, set $x^2 = y$.
  - The roots of $2y^2 + y - 6$ are $y = 3/2$ and $y = -2$.
  - $x^2 = 3/2$ or $x^2 = -2$, and so $x = \pm\sqrt{3/2} = \pm\sqrt{6}/2$ or $x = \pm i\sqrt{2}$.
  - Hence the complete factorisation of $2x^4 + x^2 - 6$ over $\mathbb{C}$ is

$$\begin{aligned} 2x^4 + x^2 - 6 &= 2(x - \sqrt{6}/2)(x + \sqrt{6}/2)(x - i\sqrt{2})(x + i\sqrt{2}) \\ &= (\sqrt{2}\,x - \sqrt{3})(\sqrt{2}\,x + \sqrt{3})(x - i\sqrt{2})(x + i\sqrt{2}). \end{aligned}$$

  - Its complete factorisation over $\mathbb{R}$ is

$$2x^4 + x^2 - 6 = 2(x - \sqrt{6}/2)(x + \sqrt{6}/2)(x^2 + 2),$$

  - Its complete factorisation over $\mathbb{Q}$ is $2x^4 + x^2 - 6 = (2x^2 - 3)(x^2 + 2)$.

Self-reciprocal polynomials

- A polynomial of degree n is *self-reciprocal* if $x^n \cdot f(1/x) = f(x)$.
- If $f(x) = a_n x^n + \cdots + a_1 x + a_0$ then *its reciprocal polynomial* is

$$x^n \cdot f\left(\frac{1}{x}\right) = a_n + a_{n-1}x + a_{n-2}x^2 + \cdots + a_1 x^{n-1} + a_0 x^n,$$

  and so $x^n \cdot f(1/x)$ is a polynomial of degree n, whose coefficients are the coefficients of $f(x)$ read in the opposite order.
- Hence a polynomial $f(x)$ is self-reciprocal if it equals its reciprocal polynomial. This means that its sequence of coefficients reads the same backwards as forwards: $a_n = a_0$, $a_{n-1} = a_1$, etc.
- The definition of self-reciprocal shows that all roots α are nonzero (because $a_0 = a_n \neq 0$), and that whenever α is a root, its *reciprocal* $1/\alpha$ is a root as well. (This justifies their name.)

- A cubic self-reciprocal has -1 as a root, so this is easy.
- A quartic self-reciprocal has the form $ax^4 + bx^3 + cx^2 + bx + a$.
- The idea is to match x and $1/x$, by taking their sum $x + 1/x$.
- Dividing by x^2 we find $ax^2 + bx + c + b/x + a/x^2 = 0$, that is,

$$a\left(x^2 + \frac{1}{x^2}\right) + b\left(x + \frac{1}{x}\right) + c = 0.$$

- Now note that $x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2$, and so we can write our equation in the equivalent form

$$a\left(x + \frac{1}{x}\right)^2 + b\left(x + \frac{1}{x}\right) + c - 2a = 0.$$

- Now we set $y = x + 1/x$ and solve $ay^2 + by + c - 2a = 0$.
- If this has solutions β_1 and β_2, then it remains to solve

$$x + 1/x = \beta_1 \quad \text{and} \quad x + 1/x = \beta_2.$$

- Altogether we get at most four solutions of our quartic equation.

- EXAMPLE. Find all complex roots of $6x^4+5x^3-38x^2+5x+6$.
  - Equate to zero, divide by x^2, rearrange the terms, and get
$$6\left(x^2+\frac{1}{x^2}\right)+5\left(x+\frac{1}{x}\right)-38=0.$$
  - Using $(x+1/x)^2=x^2+2+1/x^2$ the equation becomes
$$6\left(x+\frac{1}{x}\right)^2+5\left(x+\frac{1}{x}\right)-50=0,$$
which reads $6y^2+5y-50=0$ after setting $x+1/x=y$.
  - Roots are $5/2$ and $-10/3$, so $x+1/x=5/2$ or $x+1/x=-10/3$.
  - Hence $2x^2-5x+2=0$ or $3x^2+10x+3=0$. Now solve these.
  - We find the roots of the original polynomial: 2, $1/2$, -3, $-1/3$.
  - Hence
$$\begin{aligned}6x^4+5x^3-38x^2+5x+6&=6(x-2)(x-1/2)(x+3)(x+1/3)\\&=(x-2)(2x-1)(x+3)(3x+1).\end{aligned}$$
  - Because the roots have turned out to be all rational in this example, we could of course also have found them by the Rational Root Test.

Square roots of complex numbers

- If $a + ib = r\exp(i\theta) = r(\cos\theta + i\sin\theta)$, then its square roots are

$$\pm\sqrt{r}\exp(i\theta/2) = \pm\sqrt{r}\cdot\big(\cos(\theta/2) + i\sin(\theta/2)\big).$$

- We may use $\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos\theta}{2}}$ and $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}}$ etc., but we would like to find the square roots $x + iy$ and $-x - iy$ of $a + ib \neq 0$ algebraically, without using trigonometric functions.
- We have $(x + iy)^2 = a + ib$, that is, $x^2 - y^2 + 2ixy = a + ib$.
- Because $x, y, a, b \in \mathbb{R}$, this is equivalent to $\begin{cases} x^2 - y^2 = a \\ 2xy = b \end{cases}$
- Assuming $b \neq 0$ (otherwise $a + ib \in \mathbb{R}$, easy), second eq. implies $xy \neq 0$, so $x \neq 0$. Dividing by x we find $y = b/(2x)$.
- Substituting into first equation we get $x^2 - \left(\frac{b}{2x}\right)^2 = a$, and so $4x^4 - 4ax^2 - b^2 = 0$, biquadratic, which we know how to solve.

- EXAMPLE. Express the square roots of $-3+4i$ using only square roots of real numbers. (Writing $\pm\sqrt{-3+4i}$ is not an answer.)
  - We look for $x, y \in \mathbb{R}$ such that $(x+iy)^2 = -3+4i$, that is,

$$x^2 - y^2 + 2ixy = -3+4i.$$

  - Because $x, y \in \mathbb{R}$ this is equivalent to the system $\begin{cases} x^2 - y^2 = -3 \\ 2xy = 4 \end{cases}$
  - From second equation we get $y = 2/x$, and substituting into first eq. we get $x^2 - (2/x)^2 = -3$, that is, $x^4 + 3x^2 - 4 = 0$.
  - This biquadratic is easy to factorise: $(x^2-1)(x^2+4) = 0$.
  - Its roots are $x = \pm 1, \pm 2i$, but because $x \in \mathbb{R}$ we can only accept $x = \pm 1$, and then we find $y = \pm 2$ (with matching signs).
  - Hence the square roots of $-3+4i$ are $1+2i$ and $-1-2i$.
  - If we had chosen $x = \pm 2i$ instead, and correspondingly $y = \mp i$, we would still get the correct square roots $2i + i(-i) = 1+2i$ and $-1-2i$, although the procedure would be wrong (as $x \notin \mathbb{R}$).

Relations between roots and coeffs of a quadratic pol

- If a quadratic polynomial has roots α and β (in a field F), then

$$ax^2 + bx + c = a(x - \alpha)(x - \beta) = a\big(x^2 - (\alpha + \beta)x + \alpha\beta\big),$$

  and so
  - $-b/a = \alpha + \beta$ (the sum of the roots), and
  - $c/a = \alpha\beta$ (the product of the roots).
- EXAMPLE. Solve the system $\begin{cases} x + y = s \\ xy = p \end{cases}$
  - x and y will be the roots of the polynomial $(z - x)(z - y)$ in the indeterminate z, which $= z^2 - (x + y)z + xy$, and $= z^2 - sz + p$.
  - So its roots are given by the formula $\big(s \pm \sqrt{s^2 - 4p}\big)/2$.
  - One of the roots will be x, and the other will be y.
  - If the roots are distinct there are two ways to match x and y to the two roots, which gives two solutions (x, y) for our system.
  - If $s^2 = 4p$, the system has a 'double' solution $(x, y) = (s/2, s/2)$.

Symmetric systems

- If a system of two equations in x and y does not change when we interchange x and y (a symmetric system) we may try to solve it by first expressing everything in terms of $x + y$ and xy.
- EXAMPLE. Find all the complex solutions of $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = 1 \\ x + y = 2 \end{cases}$
  - LHS of first equation is $\frac{x+y}{xy}$, multiply by xy and get $\begin{cases} x + y = xy \\ x + y = 2 \end{cases}$
  - Substitute second equation in first equation: $\begin{cases} xy = 2 \\ x + y = 2 \end{cases}$
  - Because according to the first equation xy is nonzero, we did not introduce any more solutions when we multiplied by xy, and so this system is equivalent to the original system.
  - Solving this we find the solutions $(x, y) = (1 + i, 1 - i), \ (1 - i, 1 + i)$.

- EXAMPLE. Find all the complex solutions of $\begin{cases} x^2+y^2=10 \\ xy=3 \end{cases}$
  - Because $x^2+y^2=(x+y)^2-2xy$ we get $\begin{cases} (x+y)^2-2xy=10 \\ xy=3 \end{cases}$
  - Substituting second eq. into first we get $\begin{cases} (x+y)^2=16 \\ xy=3 \end{cases}$
  - This is equivalent to

$$\begin{cases} x+y=4 \\ xy=3 \end{cases} \quad \text{or} \quad \begin{cases} x+y=-4 \\ xy=3 \end{cases}$$

  - Solving those two systems as usual we find that the solutions of our original system are

$$(x,y)=(1,3),\ (3,1),\ (-1,-3),\ (-3,-1).$$